Geometric Structures on Manifolds.

- Elements of geometry
- Model spaces (E^n, S^n, H^n)
- Manifolds
- Geometric structures on manifolds.
- Gauss – Bonnet Theorem
- Orbit spaces, side-pairing.
- More (3-dimensional case, projective structures, Moduli spaces, etc.)

• Elements of geometry.

Metric Spaces

A metric space (X,d) is a set X with a metric $d: X \times X \to \mathbb{R}$ having the following properties :

(1) $d(x,y) \geq 0$, $\forall x,y \in X$, $d(x,y)=0$ iff $x=y$.

(2) $d(x,y) = d(y,x)$

(3) $d(x,z) \leq d(x,y) + d(y,z)$.

Isometries

Let $(X, d_X), (Y, d_Y)$: metric spaces.

A function $f: X \to Y$ is an "isometry"

$:\Leftrightarrow$ f is bijective & $d_Y(f(x), f(y)) = d_X(x,y)$.

The set

$$\mathrm{Isom}(X) := \{ f: X \to X \mid f \text{ is an isometry} \}$$

forms a group, whose operation is the composition.

$\mathrm{Isom}(X)$ can be given a topology, which is called the "compact-open topology". It is the topology on $\mathrm{Isom}(X)$ whose subbasis consists of the sets of the following form :

$$S(C,U) := \{ f \in \mathrm{Isom}(X) : f(C) \subset U \},$$

C: a compact subspaces of X, U: an open subset of X.

$\therefore$ $\mathrm{Isom}(X)$ is both a group and a topological space.

Geodesics

A map $f: [a,b] \to X$ is called a "geodesic segment" if $f: [a,b] \to f([a,b])$ is an isometry.

(i.e. $d(f(s), f(t)) = |s-t|$, $\forall s,t \in [0,1]$).

A map $f: \mathbb{R} \to X$ is a "geodesic"

$:\Leftrightarrow$ for each $x \in \mathbb{R}$, there are $s, t \in \mathbb{R}$ with $s < x < t$ such that $f|_{[s,t]}$ is a geodesic segment.

Volume

If (X, d) is a metric space with a measure μ, then for each measurable subset A of X, the volume of A is

$$\mathrm{Vol}(A) = \mu(A) = \int_A 1 \, d\mu .$$

Curvature

The definition of curvatures involves some technical notions coming from Riemannian geometry, and it will not be given here.

Basically, if (X, d) is a Riemannian manifold, then the curvature measures how much X is bent at each point.

- Model Spaces. (E^n, S^n, H^n).

I. The Euclidean Space E^n

E^n is the set $\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$ together with the metric

$$d(x,y) = \|x-y\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$$

for $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

Isometries

Typical examples of isometries of E^n are as follows:

(1) Translations : for $a \in E^n$, the "translation" $T_a : E^n \to E^n$ by a is the map given by

$$T_a(x) = x + a \quad , \text{ for } x \in E^n.$$

(2) Orthogonal transformations: an orthogonal transformation is a linear transformation $A : E^n \to E^n$ preserving dot products, i.e.

$$Ax \cdot Ay = x \cdot y \quad \text{for } x, y \in E^n.$$ //.

Actually, the above two examples can build every element of $\mathrm{Isom}(E^n)$. For every $f \in \mathrm{Isom}(E^n)$, there are unique orthogonal transformation $A: E^n \to E^n$ and an element $a \in E^n$ such that

$$f(x) = Ax + a \quad \text{for } x \in E^n.$$

(A: the orthogonal part of f, T_a : the translation part of f).

This shows that

$$\mathrm{Isom}(E^n) \cong O(n) \times E^n \quad (\text{homeomorphism}).$$

$$(\; f \longmapsto (\text{the orthogonal part of } f, \text{ the translation part of } f) \;)$$

Here $O(n) = \{ A \mid A \text{ is an orthogonal transformation } E^n \to E^n \}$.

Geodesics

The geodesics of E^n are straight lines :

$$\gamma(t) = ta + b \quad , t \in \mathbb{R}$$

for some $a, b \in E^n$ with $a \neq 0$.

Volumes.

The volumes of (measurable) subsets of E^n are given by their Lebesgue measures.

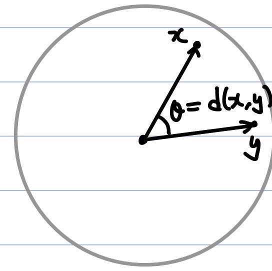
Curvatures

E^n is actually a Riemannian manifold , and the curvature of E^n at each of its points is zero .

II. The sphere S^n

For $n \in \{1, 2, \cdots\}$, the sphere S^n is the set

$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\|^2 = x \cdot x = 1 \},$$



whose metric d is given by

$$d(x,y) := \text{the angle between the unit vectors } x, y$$

$$= \cos^{-1}\left(\frac{x \cdot y}{\|x\| \|y\|} \right) = \cos^{-1}(x \cdot y) \ (\in [0, \pi])$$

Isometries

Note that if $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, then

$$d(Ax, Ay) = \cos^{-1}(Ax \cdot Ay) = \cos^{-1}(x \cdot y), \text{ for } x, y \in S^n,$$

so orthogonal transformations are isometries of S^n

$(O(n+1) \subset \mathrm{Isom}(S^n))$

Indeed, we have $\mathrm{Isom}(S^n) = O(n+1)$.

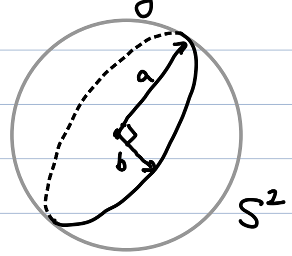
Geodesics.

Each geodesic $\gamma : \mathbb{R} \to S^n$ is of the form

$$\gamma(t) = (\cos t)\, a + (\sin t)\, b, \quad t \in \mathbb{R}$$

for some fixed $a, b \in S^n$ such that $a \cdot b = 0$.

$\to$ each geodesic of S^n is a great circle of S^n



Volumes.

The volumes of subsets of S^n is computed via a change of coordinates.

Let $g : [0,\pi]^{n-1} \times [0, 2\pi] \longrightarrow S^n$ by the map given by

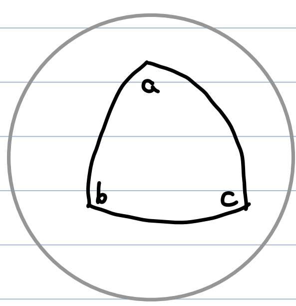
$$g(\theta_1, \cdots, \theta_n) = (x_1, \cdots, x_{n+1}) \quad \text{where}$$

$$\begin{aligned}
x_1 &= \cos\theta_1 \\
x_2 &= \sin\theta_1 \cos\theta_2 \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\vdots \\
x_n &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} \cos\theta_n \\
x_{n+1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} \sin\theta_n .
\end{aligned}$$

For a (measurable) subset X of S^n, the volume of X is given by

$$\mathrm{Vol}(X) = \int_{g^{-1}(X)} \sin^{n-1}\theta_1 \sin^{n-2}\theta_2 \cdots \sin\theta_{n-1} \, d\theta_1 \cdots d\theta_n .$$

If $\Delta = \Delta(a,b,c)$ is a triangle in S^n with angles a, b, c, then
$\mathrm{Vol}(\Delta) = a + b + c - \pi$



Curvature

S^n is a Riemannian manifold and it has a constant curvature, which is 1.

III. Hyperbolic space H^n.

Define a bilinear form on $\mathbb{R}^{n+1}$ by

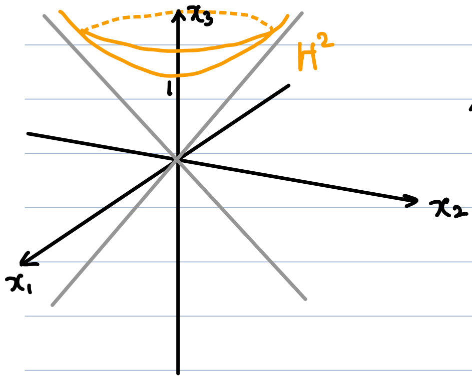
$$x \circ y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

for $x = (x_1, \cdots, x_{n+1})$, $y = (y_1, \cdots, y_{n+1}) \in \mathbb{R}^{n+1}$.

$\circ$ is called the "Lorentzian inner product".

The hyperbolic space H^n is

$$H^n = \{ x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x \circ x = -1,\ x_{n+1} > 0 \}$$



, whose metric is given by

$$d(x,y) = \cosh^{-1}(-x \circ y),$$

for $x, y \in H^n$.

Isometries

We have $\mathrm{Isom}(H^n) = PO(n,1)$

$$:= \{ A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \mid A \text{ is linear, } Ax \circ Ay = x \circ y,\ \forall x, y \}$$

$$= \{ A \in \underbrace{M(n+1, \mathbb{R})}_{(n+1)\times(n+1) \text{ matrices}} \mid A^T J A = J \}$$

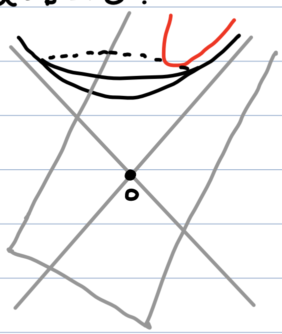
where $J = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$

Geodesics

Each geodesic $\gamma : \mathbb{R} \to H^n$ is of the form

$$\gamma(t) = (\cosh t)\, a + (\sinh t)\, b$$

for some fixed $a, b \in \mathbb{R}^{n+1}$ with $a \circ a = -1$, $b \circ b = 1$, $a \circ b = 0$.



Each geodesic is obtained by intersecting H^n with a 2-dimensional subspace of $\mathbb{R}^{n+1}$.

Volumes

The hyperbolic volume can be obtained by a change of coordinates as in the sphere.

The volume of a triangle $\Delta = \Delta(a,b,c)$ is given by

$$\mathrm{Vol}(\Delta) = \pi - (a+b+c).$$

Curvatures

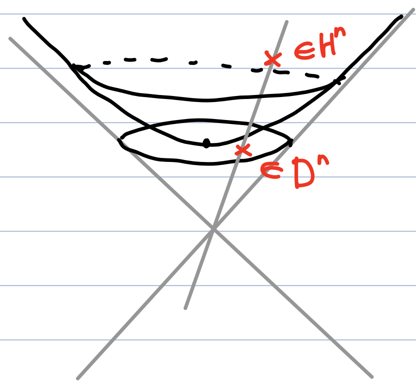
H^n also has a constant curvature, and it is -1.

Two models of H^n.

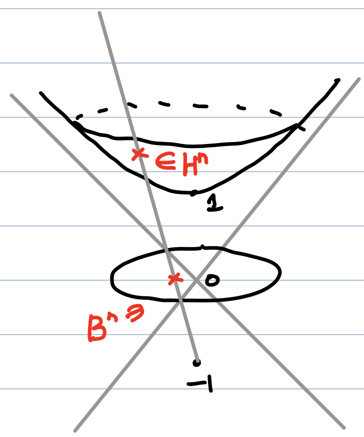
Let $D^n = \{(x_1, \cdots, x_{n+1}) \mid x_1^2 + \cdots + x_n^2 < 1,\ x_{n+1} = 1\}$,
$B^n = \{(x_1, \cdots, x_{n+1}) \mid x_1^2 + \cdots + x_n^2 < 1,\ x_{n+1} = 0\}$.

We can consider the following bijections

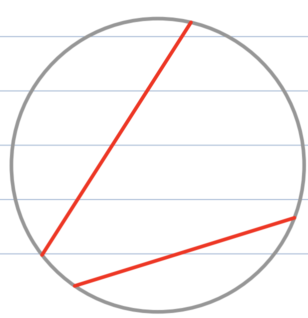
(i)

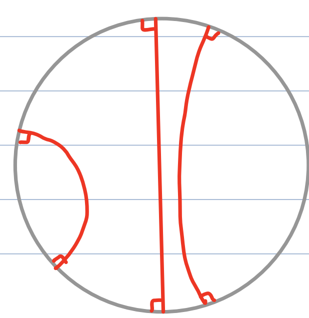


(ii)



The bijections $H^n \to D^n$, $H^n \to B^n$ makes D^n, B^n metric spaces which are isometric to H^n.
The resulting hyperbolic spaces D^n, B^n are called the "projective disk model (= Beltrami – Klein model)", "conformal disk model (Poincaré disk model)".



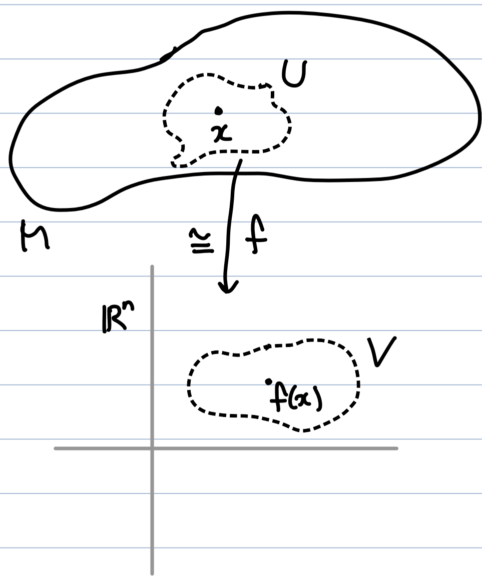


• Manifolds.

A topological space M is called an "n-dimensional manifold without boundary"
if the following conditions are met.

(i) M is second countable, Hausdorff.

(ii) For each $x \in M$, their is are open subsets $U \subset M$, $V \subset \mathbb{R}^n$ with $x \in U$
and a homeomorphism $f : U \longrightarrow V$



If M is a connected 2-dimensional manifold without boundary, then M is called a "surface".

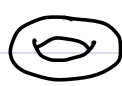
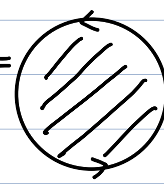
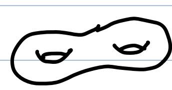
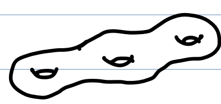
If M is a compact manifold without boundary, then M is said to be "closed".

Note that $\mathbb{R}^n$ can be replaced by S^n or H^n.

Classification of closed surfaces.

$$\chi(M) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank}(H_i(M)).$$

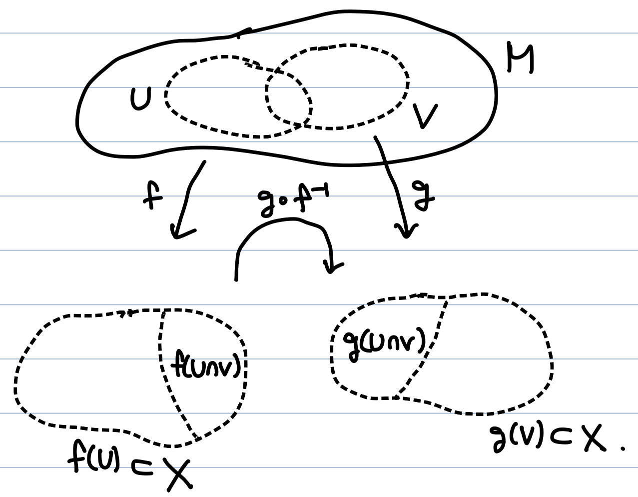
Every closed surface is homeomorphic to one of the following.

M	$\chi(M)$	M	$\chi(M)$	M	$\chi(M)$
S^2	$\boxed{2}$	$T^2 =$ 	0	$\mathbb{R}P^2 =$ 	1
		$T^2 \# T^2$ 	-2	$\mathbb{R}P^2 \# \mathbb{R}P^2$	0
		$T^2 \# T^2 \# T^2$ 	-4	$\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$	-1
		$\vdots$	$\vdots$	$\vdots$	$\vdots$

• Geometric Structures on Manifolds.

Let M : an n-manifold without boundary.
Let X be one of E^n, S^n, H^n.

A "X-chart" (f, U) is a homeomorphism $f: U \to f(U)$ such that U: open in M, $f(U)$: open in X.
We will say that two charts $(f,U), (g,V)$ are "X-compatible" if the map $g \circ f^{-1} : f(U \cap V) \to g(U \cap V)$ is a restriction of an isometry $X \to X$.



Let $\mathcal{A}$ be a collection of charts of M.
$\mathcal{A}$ is an "$(X, \mathrm{Isom}(X))$ atlas" if
(1) the domains of charts in $\mathcal{A}$ cover M
(2) Each pair of charts in $\mathcal{A}$ are X-compatible.

If $\mathcal{A}$ is given, then there is a maximal $(X, \mathrm{Isom}(X))$-atlas Φ on M such that $\mathcal{A} \subset \Phi$.
Φ is called a Euclidean (resp. spherical, hyperbolic) structure on M if $X = E^n$ (resp. $X = S^n$, $X = H^n$), and (M, Φ) is called a Euclidean (resp. spherical, hyperbolic) manifold.

In general, given a manifold M, it is not easy to show that M admits such structures.

The following theorem gives a necessary condition.

- <u>The Gauss-Bonnet theorem.</u> (application)

<u>Theorem</u> (Gauss-Bonnet).
Let M be a closed surface, which is also a Riemannian manifold with curvature $\kappa : M \to \mathbb{R}$. Then we have

$$\int_M \kappa \, dA = 2\pi \chi(M).$$

//

Now, suppose that M admits a Euclidean (resp. spherical, hyperbolic) structure.

$\to$ M is a Riemannian manifold with constant curvature $\kappa = 0$ (resp. 1, -1).

$\to$ $2\pi \chi(M) = \int_M \kappa \, dA = \kappa \operatorname{Vol}(M)$

In particular, we have $\chi(M) = 0$ (resp. $\chi(M) > 0$, $\chi(M) < 0$).

Therefore, we see that

(i) if $\chi(M) > 0$, then M can admit only spherical structures.
(ii) if $\chi(M) = 0$, then M can admit only Euclidean structures.
(iii) if $\chi(M) < 0$, then M can admit only hyperbolic structures.

• Orbit spaces / Side-pairing.

Let X be one of E^n, S^n, H^n.
Let G be a subgroup of $\mathrm{Isom}(X)$.
For $x \in X$, the "G-orbit" Gx of x is the set

$Gx = \{ gx \mid g \in G \}$. $\to$ $\{ Gx \mid x \in X \}$ form a partition of X.

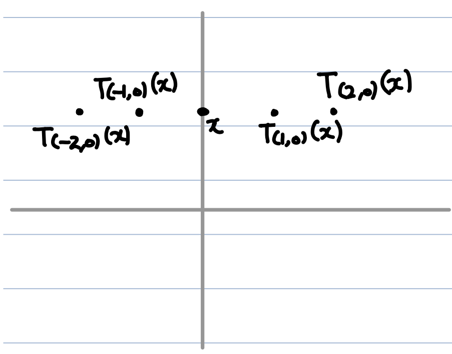
The "orbit space" is the quotient space

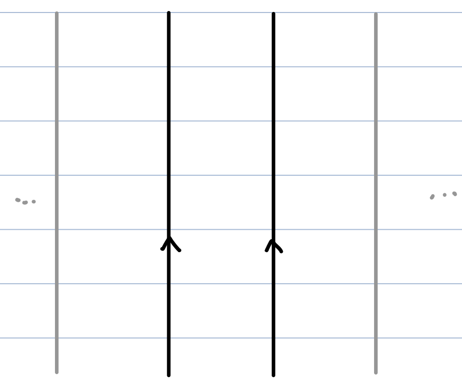
$X/G = \{ Gx \mid x \in X \}$.

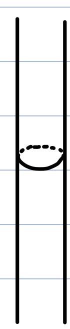
Example

Let $G = \{ T_{(j,0)} : E^n \to E^n \mid j \in \mathbb{Z} \}$

$\to$ G is an infinite cyclic subgroup of $\mathrm{Isom}(E^n)$.
The orbit space E^n/G is homeomorphic to $S^1 \times \mathbb{R}$.





$\cong$  $\cong E^n/G$.

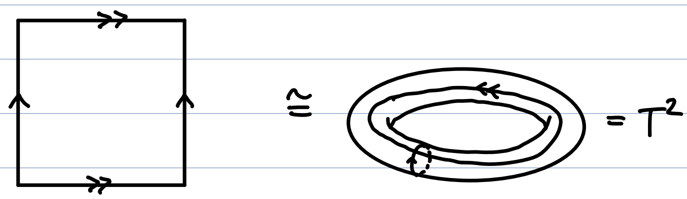
Theorem

Let $X = E^n, S^n$ or H^n, and let Γ be a discrete subgroup of $\mathrm{Isom}(X)$ which acts on X freely.

$\Rightarrow$ the orbit space X/Γ is an n-manifold which admits a $(X, \mathrm{Isom}(X))$ structure so that the projection is a local isometry.

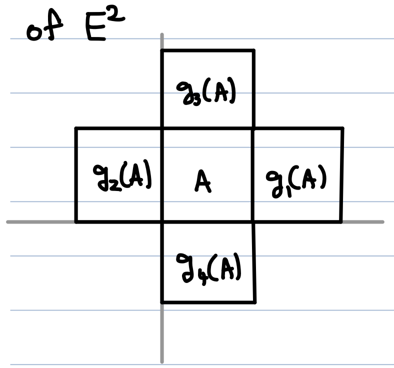
Side pairing

Let us consider the simplest case : $M = T^2$.
Recall that T^2 is obtained by the following identification :



We view T^2 in a slightly different way.

Let $A = [0,1]^2 \subset E^2$.
Consider the four isometries $g_1 = T_{(1,0)}$, $g_2 = T_{(-1,0)}$, $g_3 = T_{(0,1)}$, $g_4 = T_{(0,-1)}$.
of E^2



Let Γ be the subgroup of $\mathrm{Isom}(E^2)$ generated by g_1, g_2, g_3, g_4.
$\Rightarrow \Gamma = \{ T_{(m,n)} \mid m, n \in \mathbb{Z} \}$ $(\cong \mathbb{Z}^2)$.
$\Rightarrow \Gamma$ is a discrete subgroup of $\mathrm{Isom}(E^2)$ acting freely on E^2.
$\Rightarrow$ The orbit space E^2/Γ has a Euclidean structure.

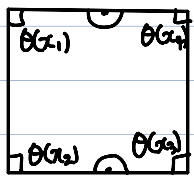
Note that $E^2/\Gamma \cong T^2$.
$\therefore$ T^2 admits a Euclidean structure.

The above technique is called "side-pairing". The idea is summarized/generalized as follows

(1) Find a polygon P in X $(= E^n, S^n$ or $H^n)$ such that we can obtain the desired closed surface M via identification.
Let $S_1, \cdots, S_r$ be the edges (= sides) of P.
(2) Find $g_{S_1}, \cdots, g_{S_r} \in \mathrm{Isom}(X)$ such that
(i) $g_{S_i}^{-1} = g_{S_j}$ for each pair S_i, S_j which we want to glue together.
(ii) $P \cap g_{S_i}(P) = S_i$

Let Γ be the subgroup of $\mathrm{Isom}(X)$ generated by $g_{s_1}, \cdots, g_{s_r}$.
Then we have the following result.

<u>Theorem</u> Suppose that Γ satisfies the following condition:
For each fixed $x \in P$, consider its orbit $P \cap \Gamma x$ in P.
Write $P \cap \Gamma x = \{x_1, \cdots, x_k\}$. Then $\theta(x_1) + \cdots + \theta(x_k) = 2\pi$.

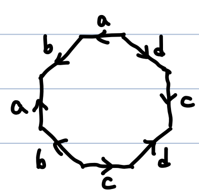


If this condition is satisfied for each $x \in P$, then the resulting orbit space X/Γ is homeomorphic to M and admits the geometric structure so that the composition $P \hookrightarrow X \to X/\Gamma$ is a local isometry.

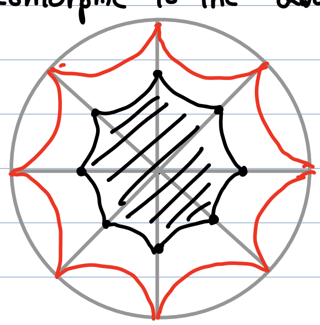
<u>Example</u>

(1) Let $\alpha : S^n \longrightarrow S^n$ be the antipodal map : $\alpha(x) = -x$.
$\rightarrow$ $\Gamma = \{id, \alpha\}$ is a discrete subgroup of $\mathrm{Isom}(S^n) = O(n+1)$.
Note that $S^n/\Gamma = \mathbb{R}P^n$
$\therefore$ $\mathbb{R}P^n$ admits a spherical structure

(2). Let $M = T^2 \# \cdots \# T^2$ (n-times, $n \geq 2$).
$\rightarrow$ M is homeomorphic to the following quotient space :



The sum of the (Euclidean) angle is $\frac{3\pi}{4} \times 8 = 6\pi \neq 2\pi$, so we consider the following $4n$-gon homeomorphic to the above $4n$-gon.



We can scale P in $H^2 (= B^2)$ so that the sum of angles is 2π.

Applying the above side-pairing process by finding appropriate isometries $g_{s_1}, \cdots, g_{s_{4n}}$, we conclude that M admits a hyperbolic structure.