Chapter 1 ■ Simplices.

Definition ■ [Geometrically independent].

Given a set $\{a_0, \cdots, a_n\}$ of points of $\mathbb{R}^N$, this set is said to be geometrically independent if for any real t_i,

$$\sum_{i=0}^{n} t_i = 0 \text{ and } \sum_{i=0}^{n} t_i a_i = 0 \text{ imply } t_0 = t_1 = \cdots = t_n = 0.$$

or equivalently, $a_1 - a_0, \cdots, a_n - a_0$ are linearly independent.

Definition ■ [n-simplex].

Let $\{a_0, \cdots, a_n\}$ be a geometrically independent set in $\mathbb{R}^N$.

Define the n-simplex σ to be the set of all points x of $\mathbb{R}^N$.

such that $x = \sum_{i=0}^{n} t_i a_i$, where $\sum_{i=0}^{n} t_i = 1$ and $t_i \geq 0$ for all i.

$\hookrightarrow$ barycentric coordinates. of x.

Example).

0-simplex : a vertex.

1-simplex : a line segment.

2-simplex : a triangle.

3-simplex : a tetrahedron.

Definition ■

The points $a_0, \cdots, a_n$ that span σ are called the vertices of σ; the number n is called. a dimension of σ.

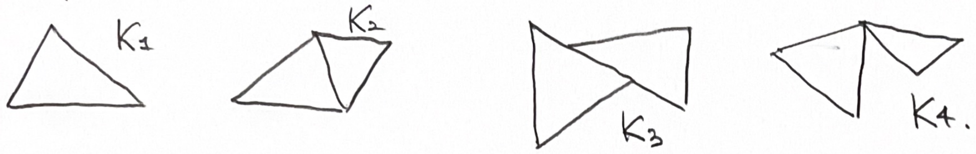
Any simplex spanned by a subset of $\{a_0, \cdots, a_n\}$ is called a face of σ; the faces of σ different from σ itself are called the proper faces. of σ; their union is called the boundary of σ denoted by $\operatorname{Bd}\sigma$.

The interior of σ is defined by $\operatorname{Int}\sigma = \sigma - \operatorname{Bd}\sigma$.

Chapter 2 ■ Simplicial Complexes

Definition ■ [Simplicial complex].
A simplicial complex K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$.
such that
- Every face of a simplex of K is in K.
- The intersection of any two simplices of K is a face of each of them.

Example)

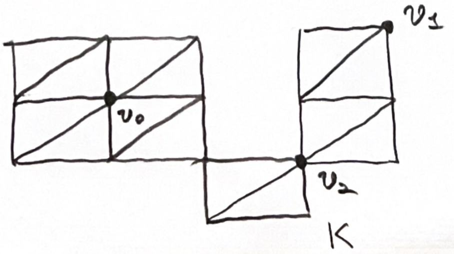


Definition ■ [Subcomplex]
If L is a subcollection of K that contains all faces of its elements, it is called a subcomplex of K.
One subcomplex of K is the collection of all simplices of K of dimension at most p; it is called the p-skeleton of K denoted by $K^{(p)}$.
Then, $K^{(0)}$ are called vertices of K.

Definition ■ [Underlying space].
Let $|K|$ denote the union of the simplices of K.
Giving each simplex of K its natural topology as a subspace of $\mathbb{R}^N$,
topologize $|K|$ by declaring a subset A of $|K|$ to be closed in $|K|$ if and only if
$A \cap \sigma$ is closed in σ, for all σ in K
The topological space $|K|$ is called the underlying space of K, or the polytope of K.

Definition ■ [Star, Closed star, Link].
If v is a vertex of K, the star of v in K, denoted by $\mathrm{St}\,v$, is the union of the interiors of those simplices of K that has v as a vertex.
Its closure, denoted by $\overline{\mathrm{St}}\,v$, is called the closed star of v in K.
The Link of v in K, denoted by $\mathrm{Lk}\,v$, is defined by $\mathrm{Lk}\,v = \overline{\mathrm{St}}\,v - \mathrm{St}\,v$.

Example)



Chapter 3 Abstract Simplicial Complexes.

Definition [Abstract simplicial complex].
An abstract simplicial complex is a collection $\mathcal{S}$ of finite non-empty sets, such that if A is an element of $\mathcal{S}$, so is every nonempty subset of A.

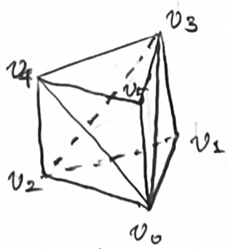
The element A is called a simplex of $\mathcal{S}$; its dimension is $|A| - 1$.
Each nonempty subset of A is called a face of A.
The dimension of $\mathcal{S}$ is the largest dimension of its simplices, or is infinite.
The vertex set V of $\mathcal{S}$ is the union of one point elements of $\mathcal{S}$.
A subcollection of $\mathcal{S}$ that is itself a complex is called a subcomplex of $\mathcal{S}$.

Definition [Vertex scheme].
If K is a simplicial complex, let V be the vertex set of K.
Let $\mathcal{K}$ be the collection of all subsets $\{a_0, \cdots, a_n\}$ of V, such that $\{a_0, \cdots, a_n\}$ span a simplex of K. The collection $\mathcal{K}$ is called the vertex scheme of K.

Definition [Geometric realization].
If the abstract simplicial complex $\mathcal{S}$ is isomorphic with the vertex scheme of the simplicial complex K, we call K a geometric realization of $\mathcal{S}$.

Example).
$\mathcal{S}$ is the collection of $\{a, f, d\}$, $\{a, b, d\}$, $\{b, c, d\}$, $\{c, d, e\}$, $\{a, c, e\}$, $\{a, e, f\}$, along with their nonempty subsets.

K is the simplicial complex pictured
$\mathcal{K}$ denote the vertex scheme of K.



Then, $f : V_{\mathcal{S}} \to V_{\mathcal{K}}$ is isomorphism, i.e., K is a gemetric realization of $\mathcal{S}$.

$$\begin{aligned} a &\mapsto v_0 \\ b &\mapsto v_1 \\ c &\mapsto v_2 \\ d &\mapsto v_3 \\ e &\mapsto v_4 \\ f &\mapsto v_5 \end{aligned}$$

Definition [Labelling].

Given a finite complex L, a labelling of vertices of L is a surjective function f mapping the vertex set of L to a set called the set of labels.

Corresponding to this labelling is an abstract complex $\mathcal{S}$ whose vertices are the labels.

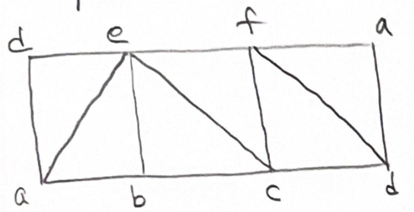
Let K be a geometric realization of $\mathcal{S}$.

Then, the vertex map from $L^{(0)}$ to $K^{(0)}$ induced by f extends to a surjective map.

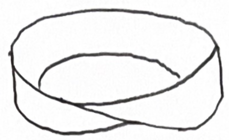
$g: |L| \to |K|$.

We say K is the complex derived from the labelled complex L, g is the associated pasting map.

Example

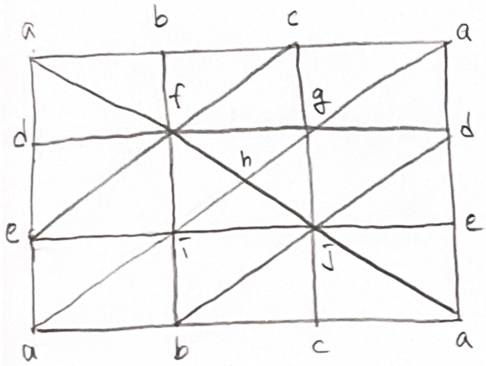


$\xrightarrow{g}$



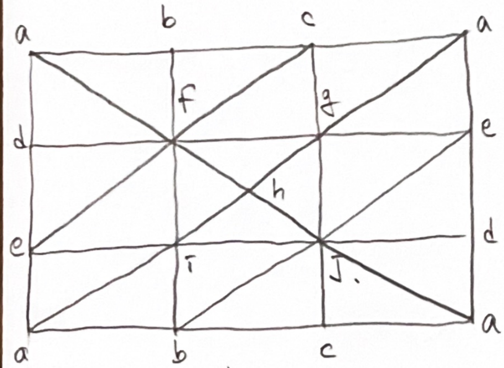
$\simeq K$.

L



$\xrightarrow{g}$

L



$\xrightarrow{g}$.

L

Chapter 4 ▪ Review of Abelian Groups.

Definition ▪ [Group].

A group is a set G with a binary operation $(*)$ on G. satisfying following condition.

- For all $a, b, c \in G$, $(a*b)*c = a*(b*c)$
- There exists an element $e \in G$. such that, for every $a \in G$, $e*a = a$, $a*e = a$.
- For each $a \in G$, there exists an element $b \in G$ such that $a*b = e$, $b*a = e$; b is unique; it is called the inverse of a

Definition ▪ [Abelian group].

If G is a group and for all $a, b \in G$, $a+b = b+a$, then G is said to be commutative, and is called abelian group.

We commonly use additive notation for abelian group.

Definition ▪ [Free abelian group].

An abelian group G is free if it has a basis

, that is, if there exists a family $\{g_\alpha\}_{\alpha \in J}$ of elements of G. such that,

for each $g \in G$ can be written uniquely as a finite sum $g = \sum_\alpha n_\alpha g_\alpha$ with $n_\alpha \in \mathbb{Z}$.

Definition ▪ [Homomorphism for groups]

A homomorphism is a map ϕ: between two groups $(G, *)$, $(G', *')$. such that.

$\phi(x*y) = \phi(x) *' \phi(y)$, for all $x, y \in G$. (preserving the operations).

Definition ▪ [Kernel/Image]

Let ϕ be a homomorphism, from G to G'.

$\text{Ker}\,\phi := \{g \in G \mid \phi(g) = e', e' \text{ is identity of } G'\}$.

$\text{Im}\,\phi := \{\phi(g) \in G' \mid g \in G\}$.

Further definitions such as subgroup, normal, factor group are omitted here, since their meanings are not that important here.

Please refer to "algebra" for the further informations.

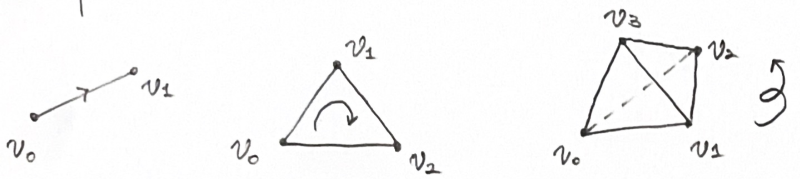
Chapter 5 Homology Groups.

Definition [Oriented simplex].

Let σ be a simplex. Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation.
If $\dim \sigma > 0$, the orderings of the vertices of σ then fall into two equivalence classes.
Each of these classes is called an orientation of σ.

We shall use the symbol $[v_0, \cdots, v_p]$ to denote oriented simplex.

Example)



$[v_0, v_1]$ $\quad$ $[v_0, v_1, v_2]$ $\quad$ $[v_0, v_1, v_2, v_3]$.

Definition [p-chain]

Let K be a simplicial complex. A p-chain on K is a function c from the set of p-simplices of K to the integers, such that:

- $c(\sigma) = -c(\sigma')$ if σ, σ' are opposite orientations of the same simplex.
- $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We define an addition of p-chains by adding their values; the resulting group is denoted by $C_p(K$ and is called group of p-chains of K. If $p<0$ or $p > \dim K$, we consider $C_p(K)$ as trivial group.

Definition [Elementary chain]

If σ is an oriented simplex, the elementary chain c corresponding to σ is the function defined as follows:

- $c(\sigma) = 1$.
- $c(\sigma') = -1$, if σ' is the opposite orientation of σ.
- $c(\tau) = 0$, otherwise.

We will use the symbol σ to denote not only oriented simplex, but also elementary p-chain.
With this convention, $\sigma' = -\sigma$.

Lemma)

$C_p(K)$ is free abelian; a basis for $C_p(K)$ can be obtained by orienting each p-simplex and using the corresponding elementary chains as basis.

Proof) Once all the p-simplices of K are oriented arbitrarily, each <u>p-chain</u> c can be written uniquely as

$$c = \sum_i n_i \cdot \sigma_i .$$

The chain c maps σ_i to n_i and all oriented p-simplices not appearing in the sum to 0.

Note)

The group $C_0(K)$ differs from the others, since 0-simplex has only one orientation.

Thus, $C_0(K)$ has a natural basis.

Definition [Boundary operator]

Define a homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$. called the boundary operator. as follows

: If $\sigma = [v_0, \cdots, v_p]$ (basis). with $p>0$, then $\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \cdot [v_0, \cdots, \hat{v}_i, \cdots, v_p]$.

Since $C_p(K) = 0$ for $p<0$, ∂_p is the trivial homomorphism for $p \leq 0$.

Lemma)

∂_p is well-defined; $\partial_p(-\sigma) = -\partial_p \sigma$.

Proof) It is sufficien to show that $\sum_{i=0}^{p} (-1)^i \cdot [v_0, \cdots, \hat{v}_i, \cdots, v_p]$ changes sign if we exchange two adjacent vertices in $[v_0, \cdots, v_p]$.

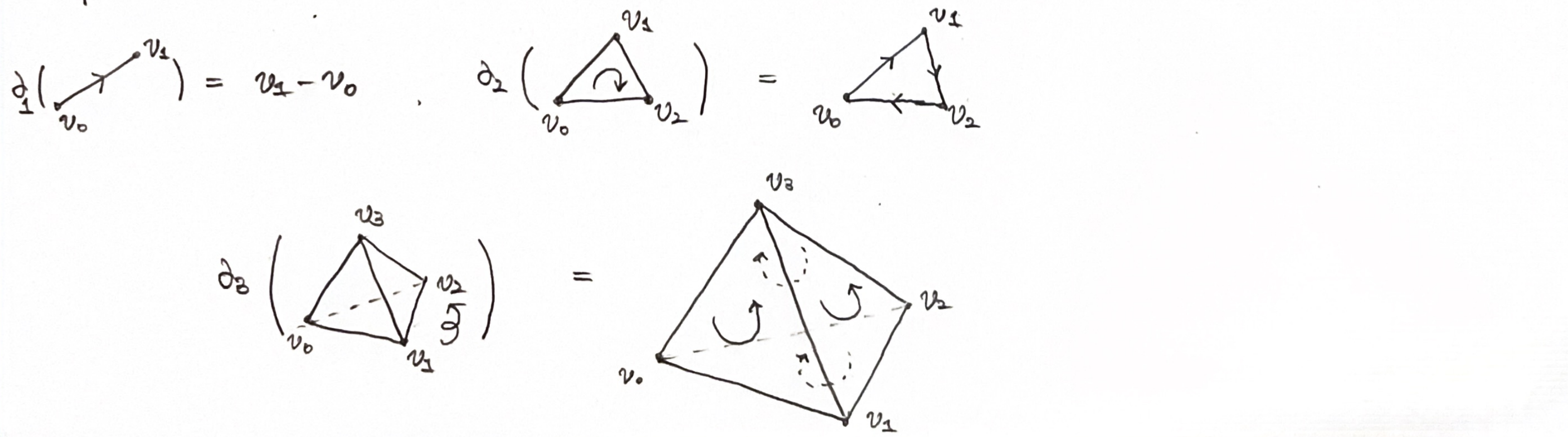
For $i \neq j, j+1$, ith terms differ by a sign.

If $i = j, j+1$, $(-1)^j [\cdots, v_{j-1}, \hat{v}_j, v_{j+1}, v_{j+2}, \cdots] + (-1)^{j+1} [\cdots, v_{j-1}, v_j, \hat{v}_{j+1}, v_{j+2}, \cdots]$ in $\partial_p [v_0, \cdots, v_j, v_{j+1}, \cdots, v_p]$.

$(-1)^j [\cdots, v_{j-1}, \hat{v}_{j+1}, v_j, v_{j+2}, \cdots] + (-1)^{j+1} [\cdots, v_{j-1}, v_{j+1}, \hat{v}_j, v_{j+2}, \cdots]$ in $\partial_p [v_0, \cdots, v_{j+1}, v_j, \cdots, v_p]$.

Thus, $\partial_p(-\sigma) = -\partial_p \sigma$.

Example).



Lemma)

$\partial_{p-1} \circ \partial_p = 0.$

Proof) $\partial_{p-1} \circ \partial_p [v_0, \cdots, v_p] = \sum_{i=0}^{p} (-1)^i \partial_{p-1} [v_0, \cdots, \hat{v}_i, \cdots, v_p]$

$$= \sum_{j<i} (-1)^i (-1)^j [\cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots] + \sum_{j>i} (-1)^i (-1)^{j-1} [\cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots] = 0.$$

Definition [p-cycles / p-boundaries].

The kernel of $\partial_p : C_p(K) \to C_{p-1}(K)$ is called the group of p-cycles and denoted by $Z_p(K)$.

The image of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ is called the group of p-boundaries and denoted by $B_p(K)$.

Lemma)

$B_p(K)$ is a normal subgroup of $Z_p(K)$.

Proof). It is sufficient to prove that $B_p(K) \subset Z_p(K)$; since $C_p(K)$ is abelian.

Suppose that $c \in B_p(K)$ for some p-chain c.

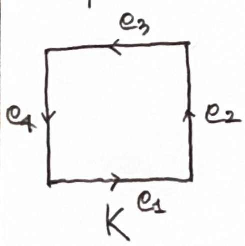
Then, there is $d \in C_{p+1}(K)$ such that $\partial_{p+1} d = c$. $\therefore$ $\partial_p c = \partial_p \cdot \partial_{p+1} d = 0$ by the preceding lemma

, i.e., $c \in \operatorname{Ker} \partial_p = Z_p(K)$.

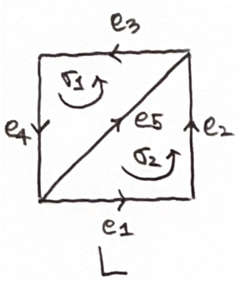
Definition [Homology group].

$H_p(K) := Z_p(K) / B_p(K)$ is called the pth homology group of K.

Example).



$H_1(K) =$



$H_1(L) =$

$H_2(L) =$

Definition. [Carry].

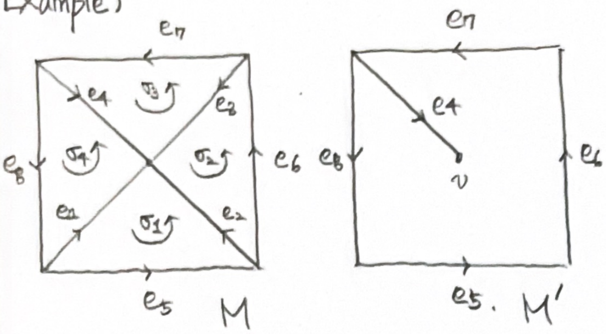
A chain c is carried by a subcomplex L of K if c has value 0 on every simplices not in L.

Definition. [Homologous]

Two p-chains c, c' are homologous if $c - c' = \partial_{p+1} d$ for some $d \in C_{p+1}(K)$

In particular, if $c = \partial_{p+1} d$, c is homologous to zero, or c bounds.

Example)



Given a 1-chain c. It is homologous to a chain that is carried by the subcomplex M' of M.

Chapter 6. Homology groups of some surfaces.

Definition. [Surface].

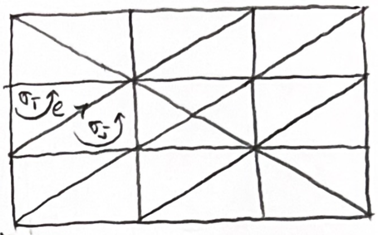
A surface is a Hausdorff space with a countable basis, each point of which has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^2$.

We shall compute the homology of compact surfaces; torus, Klein's bottle, projective plane. that can be constructed from a rectangle L by pasting its edges appropriately.

Lemma)

Let L be the complex whose underlying space is rectangle.

Let BdL denote the complex whose underlying space is the boundary of rectangle.



L

Orient each 2-simplices σ_i of L by a counterclockwise arrow.

Orient 1-simplices arbitrarily. Then:

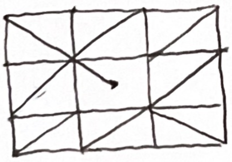
(1) Every 1-cycle of L is homologous to a 1-cycle carried by BdL.

(2) If d is a 2-chain of L and if ∂d is carried by BdL, then d is a multiple of $\sum_i \sigma_i$.

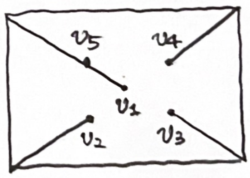
Proof). (2) is easy. If σ_i, σ_j have same edge in common, then ∂d must have 0 on the edge e.

It follows that d must have the same value on each σ_i, i.e., $d = p \cdot \sum_i \sigma_i$.

(1): By the previous example, Given a 1-chain c of L is homologous to a 1-chain c_1 carried by the subcomplex



Similarly c_1 is homologous to a 1-chain c_2 carried by the subcomplex



If c is a cycle, then c_2 is also a cycle.

It follows that c_2 must be carried by BdL, for otherwise $\partial c \neq 0$.

Theorem)

Let T denote the complex represented by the labelled rectangle L; its underlying space is Torus. Then: $H_1(T) \simeq \mathbb{Z} \times \mathbb{Z}$, $H_2(T) \simeq \mathbb{Z}$.

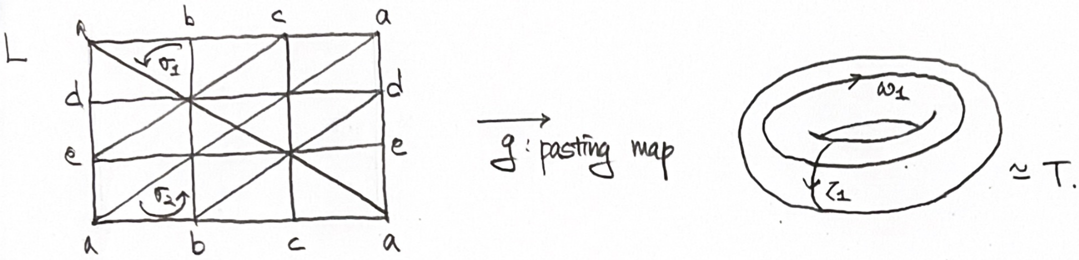
Orient each 2-simplices of L counterclockwise; use the induced orientation of the 2-simplices of T

Let $\gamma = \sum_i \sigma_i$ for each 2-simplices σ_i.

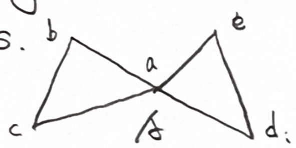
$$w_1 = [a,b] + [b,c] + [c,a]$$

$$z_1 = [a,d] + [d,e] + [e,a].$$

Then, γ generates $H_2(T)$, w_1 and z_1 represent a basis for $H_1(T)$.



Proof) Let $g: |L| \to |T|$ be the pasting map; let $A := g(|\mathrm{Bd}L|)$. Then A is homeomorphic to wedge of two circles.



Orient 1-simplices arbitrarily.

Since g affects only on $\mathrm{Bd}L$, preceeding lemma apply verbatim.

(1) Every 1-cycle of T is homologous to a 1-cycle carried by A.

(2) If d is a 2-chain of T and if ∂d is carried by A, then $d = p\gamma$ for some $p \in \mathbb{Z}$.

However two further results hold in T.

(3) If c is a 1-cycle of T carried by A, then c is of the form $nw_1 + mz_1$, $n, m \in \mathbb{Z}$.

(4) $\partial\gamma = 0$.

(3) is easy since A is the wedge of two circles.

(4): $\partial\gamma = \partial \sum_i \sigma_i$. Since each simplices for example $[a,b]$ appears in $\partial\sigma_1$ with value -1 and in $\partial\sigma_2$ with $+1$, so that $\partial\gamma$ has value 0 on $[a,b]$.

Using these results, compute the homology of T.

Every 1-cycle of T is homologous to a 1-cycle of the form $nw_1 + mz_1$. by (1), (3).

Such a cycle bounds only if it is trivial; for if $c = \partial d$ for some $d \in C_2(T)$, then by (2) $d = p\gamma$; $\partial\gamma = 0$ by (4) $c = \partial d = \partial p\gamma = p\partial\gamma = 0$. Thus $H_1(T) \simeq \mathbb{Z} \times \mathbb{Z}$. and w_1, z_1 form a basis.

Note that by (2) any 2-cycle d of T must be of the form $p\gamma$. By (4), each such 2-chain is a cycle. Since there are no 3-chains for it to bound, $H_2(T) \simeq \mathbb{Z}$ and γ is generator.

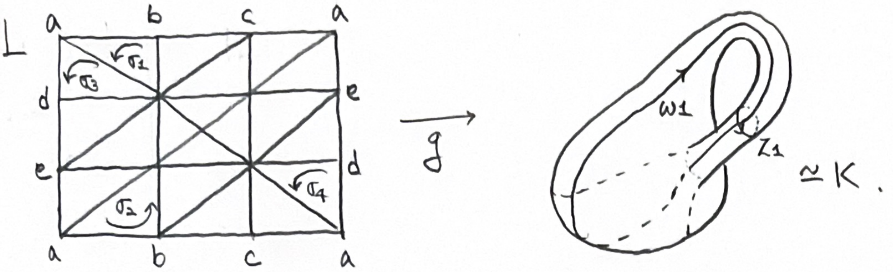
Theorem)

Let K denote the complex represented by the labelled rectangle L; its underlying space is Klein's bottle. Then: $H_1(K) \simeq \mathbb{Z} \times \mathbb{Z}/_{2\mathbb{Z}}$ and $H_2(K) = 0$.

Let $w_1 = [a,b] + [b,c] + [c,a]$

$z_1 = [a,d] + [d,e] + [e,a]$

Then, z_1 represents torsion element, and w_1 represents a generator of $H_1(K)/_T$



Proof; Let $g: |L| \to |K|$ be the pasting map; let $A := g(|BdL|)$; as before, it is the wedge of two circles.

Orient 2-simplices of K as before; let $\gamma := \sum_i \sigma_i$.

Orient 1-simplices arbitrarily.

Note that (1), (2) hold.

Because A is the wedge of two circles, (3) holds as well.

However; (4) is different; $\partial\gamma = 2z_1$.

(4): $\partial\gamma = \partial \sum_i \sigma_i$. For example, $[a,b]$ appears in $\partial\sigma_1$ with value -1 and in $\partial\sigma_2$ with $+1$, while $[a,d]$ appears in $\partial\sigma_3$ and $\partial\sigma_4$ with value $+1$.

Using these results, compute the homology of K.

Every 1-cycle of K is homologous to a cycle of the form $nw_1 + mz_1$, by (1), (3).

If $c = \partial d$ for some $d \in C_2(K)$, then $d = p\gamma$ by (2); whence $\partial d = 2pz_1$. Thus, $nw_1 + mz_1$ bounds if and only if $m = 2p$, $n = 0$.

Thus, $H_1(K) \simeq \mathbb{Z} \times \mathbb{Z}/_{2\mathbb{Z}}$.

Note that by (2) any 2-cycle d of K must be of the form $p\gamma$; since $p\gamma$ is not a cycle by (4

; $H_2(K) = 0$.

Chapter 17 Zero-dimensional Homology

We have not yet computed zero-dimensional homology $H_0(K)$. since vertex, 0-simplex is slightly different from other simplex (has only one direction).

Theorem)
Let K be a complex. Then the group $H_0(K)$ is free abelian. If $\{v_\alpha\}$ is a collection consisting of one vertex from each component of $|K|$, then the homology classes of the chains v_α form a basis for $H_0(K)$.

Proof). Step 1). If v, w are vertices in K, let us define a relation $v \sim w$ if there is a sequence $a_0, \cdots, a_n$ of vertices of K such that $v = a_0$, $w = a_n$ and $[a_i, a_{i+1}]$ is a 1-simplex of K for all i.
This relation is clearly an equivalence relation.
Given v, define. $C_v := \{\mathrm{St}\, w \mid w \sim v\}$.
We show that C_v are the components of $|K|$.
Note first that C_v is open. Furthermore, if $v \sim v'$, then $C_v = C_{v'}$.
Second, we show that C_v is path connected.
Given v, let $w \sim v$ and $x \in \mathrm{St}\, w$. Choose a sequence $a_0, \cdots, a_n$; as before.
Then, polygonal line path $a_0, \cdots, a_n, x$ lies in C_v; For $a_i \sim v$ by definition, so that $\mathrm{St}\, a_i \subset C_v$.
In particular, $[a_i, a_{i+1}]$ lies in C_v. Similarly, line segment $a_n x$ lies in $\mathrm{St}\, a_n \subset C_v$.
Hence, C_v is path connected.
Third, we show that distinct sets $C_v, C_{v'}$ are disjoint.
Suppose x is a point of $C_v \cap C_{v'}$. Then $x \in \mathrm{St}\, w$ for some w equivalent to v, and $x \in \mathrm{St}\, w'$ for some w' equivalent to v'. Since x has positive baricentric coordinates with respect to both w and w', some simplex of K has w and w' as vertices. Then, $[w, w']$ must be a 1-simplex of K, so $w \sim w'$.
It follows that $v \sim v'$, so that $C_v = C_{v'}$.
Being connected, open, and disjoint, C_v are the components of $|K|$.

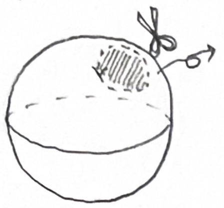
Step 2) Now we prove the theorem. Let $\{v_\alpha\}$ be a collection of vertices containing one vertex v_α from each component C_α of $|K|$. Given a vertex w of K, it belongs to some component of $|K|$, say C_α. Then, $w \sim v_\alpha$, so there is a sequence $a_0, \cdots, a_n$ of vertices as before.
The 1-chain $[a_0, a_1] + \cdots + [a_{n-1}, a_n]$ has as its boundary the 0-chain $a_n - a_0 = v_\alpha - w$. Thus, the 0-chain w is homologous to the 0-chain v_α. We conclude that every 0-chain in K is homologous to a linear combination of the 0-chains v_α.
We now show that no nontrivial chain of the form $c = \sum_\alpha n_\alpha v_\alpha$ bounds.
Suppose $c = \partial d$ for some 1-chain d. Since each 1-simplex of K lies in a unique component of $|K|$, we can write $d = \sum_\alpha d_\alpha$, where d_α consists of those terms of d that are carried by C_α.

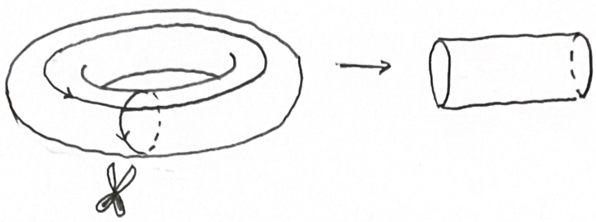
Since $\partial d = \sum_{\alpha} \partial d_{\alpha}$ and ∂d_{α} is carried by C_{α}., we conclude that $\partial d_{\alpha} = n_{\alpha} v_{\alpha}$.
It follows that $n_{\alpha} = 0$. for each α; for let $\varepsilon : C_0(K) \to \mathbb{Z}$ be the homomorphism defined by $\varepsilon(v) = 1$ for each vertex v of K. Then, $\varepsilon(\partial[v,w]) = \varepsilon(w - v) = 1 - 1 = 0$. for any 1-chain $[v,w]$.
As a result, $\varepsilon(\partial d) = \varepsilon(\sum_{\alpha} \partial d_{\alpha}) = \sum_{\alpha} \varepsilon(\partial d_{\alpha}) = \sum_{\alpha} \varepsilon(n_{\alpha} v_{\alpha}) = n_{\alpha} = 0$.

Supplement. Geometric Meaning of Homology Group for Compact Surface.

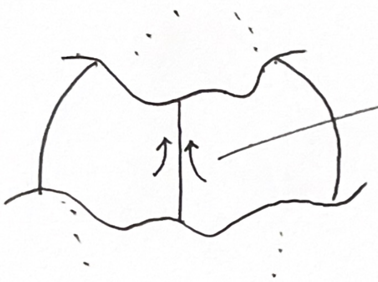
As you know, $H_0(K)$ is number of component of $|K|$.
How about $H_1(K)$ and $H_2(K)$?
Since 1-cycle can be considered as closed curve cutting line, and being factored by $B_1(K)$ means that the resulting cycle is not a boundary of some part of surface (2-chain).

Example)





2-cycle means that the orientations of each 2-simplices can be arranged to make its boundary value on each edges. Since $C_3(K)=0$, if $H_2(K)=0$, then K is not orientable at all., else K has orientable component.



Classification of compact & connected surfaces.

surface	$H_0(X)$	$H_1(X)$	$H_2(X)$
S^2	$\mathbb{Z}$	0	$\mathbb{Z}$
P^2	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
$P^2 \# P^2 \simeq K$	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	0
T^2	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$
$T^2 \# T^2$	$\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$
$\vdots$			

Thank you for comming.