바젤 문제의 여러 가지 풀이

– 바젤문제

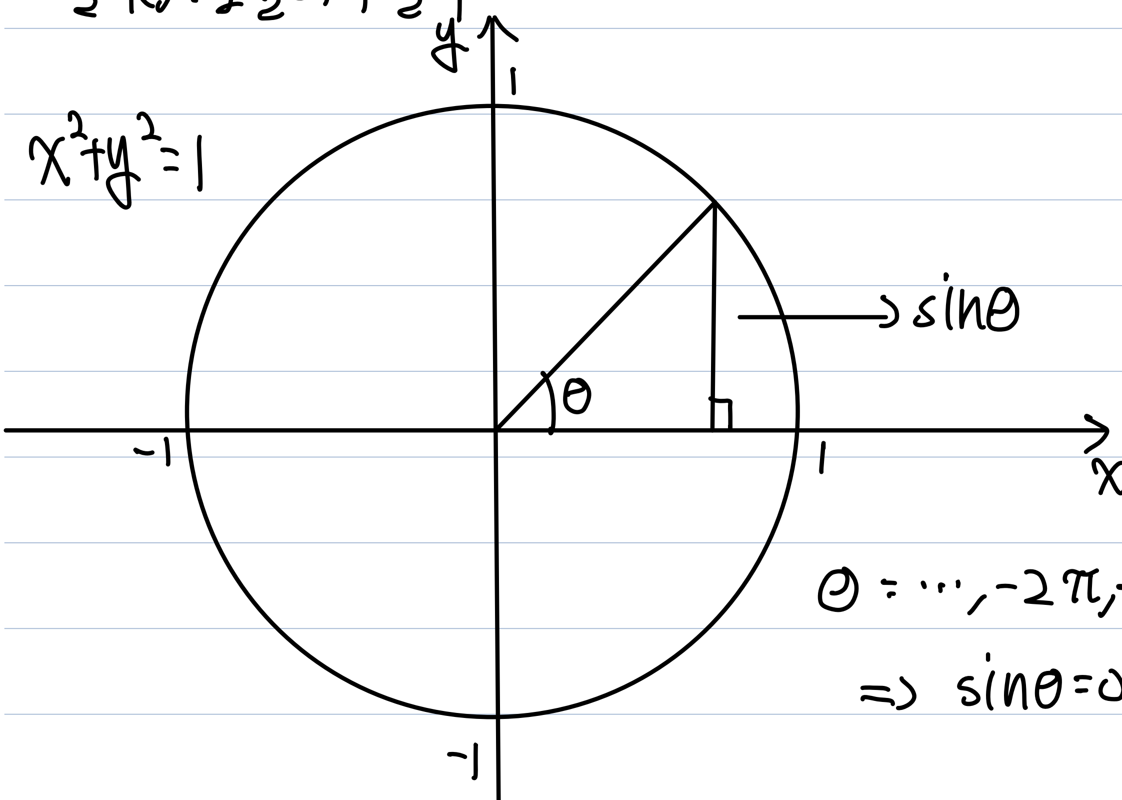
$\sum_{n=1}^{\infty}\frac{1}{n^2}$ 을 닫힌 형식으로 구하여라!

* 닫힌 형식?

$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots = 1.202\cdots$ => 닫힌 형식 X

$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots = 2$ => 닫힌 형식 O

풀이①: 오일러의 풀이



$\theta = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

$\Rightarrow \sin\theta = 0$

$\Rightarrow$ $\sin x = 0$ 의 해 : $\cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

$$\sin x = kx\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots$$

(∵ 바이어슈트라스의 곱정리)

$$\sin x = kx\prod_{n=1}^{\infty}\left(1-\frac{x}{n\pi}\right)\left(1+\frac{x}{n\pi}\right)$$

$$= kx\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

$$\frac{\sin x}{x} = k\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right) \quad (x \neq 0)$$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0} k\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

$$1 = k\prod_{n=1}^{\infty} 1 = k \quad \Rightarrow k=1$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$$

이차항계수 : $-\frac{1}{3!} = -\frac{1}{6}$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots$$

⇒ 모든 항이 상수와 이차항으로 이루어져 있음

⇒ 이차항 : 모든 상수와 모든 이차항을 곱한 것들의 합

$$-\frac{x^2}{\pi^2} - \frac{x^2}{2^2\pi^2} - \frac{x^2}{3^2\pi^2} - \cdots$$

⇒ 이차항계수 : $-\frac{1}{\pi^2} - \frac{1}{2^2\pi^2} - \frac{1}{3^2\pi^2} - \cdots = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

풀이②) 이중적분 풀이

$$I_m = \int_1^0 x^m \ln x \, dx \qquad (m = 0, 1, 2, \cdots)$$

$$I_0 = \int_1^0 \ln x \, dx = x(\ln x - 1)\Big|_1^0$$

$$= \lim_{b \to 0+} x(\ln x - 1)\Big|_1^b = \lim_{b \to 0+} \{ b(\ln b - 1) + 1 \}$$

$$= \lim_{b \to 0+} (b(\ln b) - b + 1)$$

$$\left(\lim_{b \to 0+} b \ln b = \lim_{b \to 0+} \frac{\ln b}{\frac{1}{b}} = \lim_{b \to 0+} \frac{\frac{1}{b}}{-\frac{1}{b^2}} = \lim_{b \to 0+} -b = 0 \right)$$

$$= 0 - 0 + 1 = 1$$

$$\therefore I_0 = 1$$

$$I_m = \int_1^0 x^m \ln x \, dx \qquad (m = 1, 2, 3, \cdots)$$

$$= \frac{x^{m+1}}{m+1} \ln x \Big|_1^0 - \int_1^0 \frac{x^m}{m+1} dx$$

$$= \lim_{b \to 0+} \frac{x^{m+1} \ln x}{m+1}\Big|_1^b - \frac{x^{m+1}}{(m+1)^2}\Big|_1^0$$

$$= \lim_{b \to 0+} \frac{b^{m+1} \ln b}{m+1} - \left(0 - \frac{1}{(m+1)^2} \right)$$

$$= \frac{1}{m+1} \lim_{b \to 0+} \{ \underset{\Downarrow\, 0}{b^m} \underset{\Downarrow\, 0}{(b \ln b)} \} + \frac{1}{(m+1)^2} = \frac{1}{(m+1)^2}$$

$$\therefore I_m = \frac{1}{(m+1)^2} \quad (m=1,2,3,\cdots)$$

$I_0 = 1$ 이므로 $\quad I_m = \dfrac{1}{(m+1)^2} \quad (m=0,1,2,\cdots)$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{4}{3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{4}{3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{4}{3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$= \frac{4}{3}\sum_{n=0}^{\infty} I_{2n} \qquad \because I_m = \frac{1}{(m+1)^2}$$

$$= \frac{4}{3}\sum_{n=0}^{\infty}\int_1^0 x^{2n}\ln x\,dx$$

$$= \frac{4}{3}\int_1^0 \ln x \sum_{n=0}^{\infty} x^{2n}\,dx$$

$$= \frac{4}{3}\int_1^0 (\ln x)\frac{1}{1-x^2}\,dx \qquad \because \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$$

$$(0<x<1)$$

$$= \frac{4}{3}\int_1^0 \frac{\ln x}{1-x^2}\,dx$$

$$= \frac{4}{3}\int_1^0 \frac{1}{1-x^2}\left(\frac{1}{2}\ln(x)^2\right)dx$$

$$= \frac{4}{3}\cdot\frac{1}{2}\int_1^0 \frac{1}{1-x^2}\left(\ln x^2 - \ln 1\right)dx \qquad \ln 1 = 0$$

• $\ln\left(\frac{1+x^2y^2}{1+y^2}\right) \quad y=0 \Rightarrow \ln 1$

$$\lim_{n\to\infty} \ln\left(\frac{1+x^2y^2}{1+y^2}\right) = \lim_{n\to\infty} \ln\left(\frac{\frac{1}{y^2}+x^2}{\frac{1}{y^2}+1}\right) = \ln x^2$$

$$\ln x^2 - \ln 1 = \ln\left(\frac{1+x^2y^2}{1+y^2}\right)\Bigg|_0^\infty$$

$$= \ln(1+x^2y^2) - \ln(1+y^2)\Big|_0^\infty$$

$$= \int_0^\infty \frac{2x^2y}{1+x^2y^2} - \frac{2y}{1+y^2}\,dy$$

$$= \int_0^\infty \frac{2x^2y(1+y^2) - 2y(1+x^2y^2)}{(1+x^2y^2)(1+y^2)}\,dy$$

$$2x^2y(1+y^2)-2y(1+x^2y^2)=2x^2y+2x^2y^3-2y-2x^2y^3$$
$$=2x^2y-2y$$

$$\int_0^\infty \frac{2y(x^2-1)}{(1+x^2y^2)(1+y^2)}\,dy$$

$$\frac{2}{3}\int_1^0 \frac{1}{1-x^2}(\ln x^2-\ln 1)\,dx$$

$$=\frac{2}{3}\int_1^0 \frac{1}{1-x^2}\int_0^\infty \frac{2y(x^2-1)}{(1+x^2y^2)(1+y^2)}\,dy\,dx$$

$$=\frac{2}{3}\int_0^1 \frac{2(x^2-1)}{x^2-1}\int_0^\infty \frac{y}{(1+x^2y^2)(1+y^2)}\,dy\,dx$$

$$=\frac{4}{3}\int_0^1\int_0^\infty \frac{y}{(1+x^2y^2)(1+y^2)}\,dy\,dx$$

$$=\frac{4}{3}\int_0^\infty\int_0^1 \frac{y}{(1+x^2y^2)(1+y^2)}\,dx\,dy$$

$$=\frac{4}{3}\int_0^\infty \frac{y}{1+y^2}\int_0^1 \frac{1}{1+x^2y^2}\,dx\,dy$$

$$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$

$$\int_0^1 \frac{1}{1+y^2x^2}\,dx=\frac{1}{y}\arctan yx\Big|_0^1$$
$$=\frac{1}{y}\arctan y$$

$$=\frac{4}{3}\int_0^\infty \frac{y}{1+y^2}\cdot\frac{1}{y}\arctan y\,dy$$

$$= \frac{4}{3}\int_0^\infty \frac{\arctan y}{1+y^2}\,dy$$

$$= \frac{2}{3}\int_0^\infty 2\arctan y \cdot \frac{1}{1+y^2}\,dy$$

$$= \frac{2}{3}\left(\arctan y\right)^2\Big|_0^\infty$$

$$\arctan y \xrightarrow[y\to\infty]{} \frac{\pi}{2}$$

$$= \frac{2}{3}\left\{\left(\frac{\pi}{2}\right)^2 - 0\right\} = \frac{2}{3}\cdot\frac{\pi^2}{4} = \frac{\pi^2}{6}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

풀이③ : 코시의 증명

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\theta = nx$$

$$e^{inx} = \cos nx + i\sin nx$$

$$e^{inx} = (e^{ix})^n = (\cos x + i\sin x)^n = \cos nx + i\sin nx$$

$$\frac{\cos(nx) + i\sin(nx)}{(\sin x)^n} = \frac{(\cos x + i\sin x)^n}{(\sin x)^n} = (\cot x + i)^n$$

$$(\cot x + i)^n = \sum_{r=0}^{n} \binom{n}{r} (\cot x)^{n-r} i^r$$

$i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ \cdots$

$$\sum_{r=0}^{n} \binom{n}{r} (\cot x)^{n-r} i^r = \underbrace{\sum_{r=0} \binom{n}{2r} (\cot x)^{n-2r} i^{2r}}_{\text{실수}} + \underbrace{\sum_{r=0} \binom{n}{2r+1} (\cot x)^{n-2r-1} i^{2r+1}}_{\text{허수}}$$

$$\sum_{r=0} \binom{n}{2r+1} (\cot x)^{n-2r-1} i^{2r+1}$$

$$= i \sum_{r=0} \binom{n}{2r+1} (\cot x)^{n-2r-1} i^{2r}$$

$$= i \sum_{r=0} \binom{n}{2r+1} (-1)^r (\cot x)^{n-2r-1} = i \frac{\sin(nx)}{(\sin x)^n}$$

$$\frac{\sin(nx)}{(\sin x)^n} = \sum_{r=0} \binom{n}{2r+1} (-1)^r (\cot x)^{n-2r-1}$$

$n = 2m+1,\ x = u_k \quad \left(u_k = \frac{k\pi}{2m+1}\right) \quad (k = 1, 2, 3, \cdots, m)$

$$nx = (2m+1) \cdot \frac{k\pi}{2m+1} = k\pi$$

$\sin k\pi = 0 \qquad \therefore \frac{\sin(nx)}{(\sin x)^n} = 0$

$$0 = \sum_{r=0} \binom{2m+1}{2r+1} (-1)^r (\cot u_k)^{2m+1-2r-1}$$

$$= \binom{2m+1}{1} (\cot u_k)^{2m} - \binom{2m+1}{3} (\cot u_k)^{2m-2} + \binom{2m+1}{5} (\cot u_k)^{2m-4} - \cdots$$

$$t_k = \cot^2 u_k$$

$$0 = \binom{2m+1}{1} t_k^{m} - \binom{2m+1}{3} t_k^{m-1} + \binom{2m+1}{5} t_k^{m-2} - \cdots$$

↳ 해가 $t_1, t_2, t_3, \cdots, t_m$인 방정식

$$t_1 + t_2 + t_3 + \cdots + t_m = \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{\frac{(2m+1)2m(2m-1)}{3\cdot 2\cdot 1}}{2m+1} = \frac{m(2m-1)}{3}$$

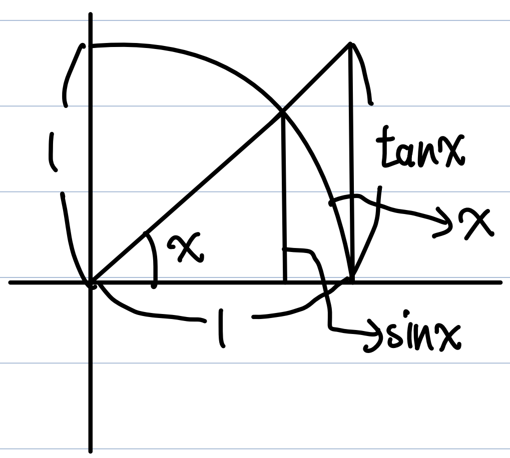
$$t_k = \cot^2 u_k$$

$$\sum_{k=1}^{m} t_k = \sum_{k=1}^{m} \cot^2 u_k = \frac{m(2m-1)}{3}$$

$$\cot^2 u_k + 1 = \csc^2 u_k \qquad \cot^2 u_k = \csc^2 u_k - 1$$

$$\sum_{k=1}^{m} (\csc^2 u_k - 1) = \frac{m(2m-1)}{3} = \sum_{k=1}^{m} \csc^2 u_k - m$$

$$\sum_{k=1}^{m} \csc^2 u_k = \frac{m(2m-1)}{3} + m \qquad \therefore \sum_{k=1}^{m} \csc^2 u_k = \frac{m(2m+2)}{3}$$



$$\sin x < x < \tan x$$

$$\cot x < \frac{1}{x} < \csc x$$

$$\cot^2 x < \frac{1}{x^2} < \csc x^2$$

$(\cot x > 0)$

$$\sum_{k=1}^{m} \cot^2 u_k < \sum_{k=1}^{m} \frac{1}{u_k^2} < \sum_{k=1}^{m} \csc u_k^2$$

$$\frac{m(2m-1)}{3} < \sum_{k=1}^{m} \frac{1}{u_k^2} < \frac{m(2m+2)}{3}$$

$$u_k : \frac{k\pi}{2m+1}$$

$$\frac{m(2m-1)}{3} < \sum_{k=1}^{m} \frac{(2m+1)^2}{k^2\pi^2} < \frac{m(2m+2)}{3}$$

$$\frac{\pi^2 m(2m-1)}{3(2m+1)^2} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{\pi^2 m(2m+2)}{3(2m+1)^2}$$

$$\lim_{m\to\infty} \frac{\pi^2 m(2m-1)}{3(2m+1)^2} = \lim_{m\to\infty} \frac{\pi^2}{3} \cdot \frac{2m^2-m}{4m^2+4m+1} = \frac{\pi^2}{3} \cdot \frac{1}{2} = \frac{\pi^2}{6}$$

$$\lim_{m\to\infty} \frac{\pi^2 m(2m+2)}{3(2m+1)^2} = \lim_{m\to\infty} \frac{\pi^2}{3} \cdot \frac{2m^2+2m}{4m^2+4m+1} = \frac{\pi^2}{3} \cdot \frac{1}{2} = \frac{\pi^2}{6}$$

$$\frac{\pi^2}{6} \le \lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{k^2} \le \frac{\pi^2}{6}$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

풀이 ④ : 로그 함수의 미분

※ 소수와 원주율

- 에라토스테네스의 체

소수만 남기는 방법

	2	3	5	7	
11		13		17	19
		23			29
31				37	
41		43		47	
		53			59
61				67	
71		73			79
81		83			89
				97	

- 제타함수

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

우변에 1만 남겨보자!

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots$$

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{15^s} + \frac{1}{17^s} + \cdots$$

$$= \left(1 - \frac{1}{2^s}\right)\zeta(s)$$

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \cdots$$

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) - \frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \cdots$$

$$= \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\zeta(s)$$

$$\vdots$$

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots\zeta(s) = 1$$

$$\prod_{p \in \text{소수}}\left(1 - \frac{1}{p^s}\right)\zeta(s) = 1$$

$$\zeta(s) = \prod_{p \in \text{소수}}\left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \in \text{소수}}\left(1 - p^{-s}\right)^{-1}$$

$$\boxed{\zeta(s) = \prod_{p \in \text{소수}}\left(1 - p^{-s}\right)^{-1}}$$

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4} p_5^{a_5} \cdots$$

$p_1, p_2, p_3, \cdots$ 는 소수 $(p_i < p_{i+1})$

a_i는 음이 아닌 정수 $\Rightarrow$ n에 따라 변하는 수

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{a_1, a_2, a_3, \cdots \geq 0} \frac{1}{p_1^{a_1 s} p_2^{a_2 s} p_3^{a_3 s} \cdots}$$

$$= \sum_{a_1=0}^{\infty} \frac{1}{p_1^{a_1 s}} \sum_{a_2=0}^{\infty} \frac{1}{p_2^{a_2 s}} \sum_{a_3=0}^{\infty} \frac{1}{p_3^{a_3 s}} \cdots$$

$$= \prod_{p\text{는 소수}} \sum_{a=0}^{\infty} \frac{1}{p^{as}} = \prod_{p\text{는 소수}} \frac{1}{1 - \frac{1}{p^s}}$$

$$= \prod_{p\text{는 소수}} (1 - p^{-s})^{-1}$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$= \prod_{p\text{는 소수}} (1 - p^{-2})^{-1} = \frac{\pi}{6}$$

$$= (1 - 2^{-2})^{-1} (1 - 3^{-2})^{-1} (1 - 5^{-2})^{-1} \cdots$$

$$= \frac{1}{1 - \frac{1}{2^2}} \cdot \frac{1}{1 - \frac{1}{3^2}} \cdot \frac{1}{1 - \frac{1}{5^2}} \cdots = \frac{\pi^2}{6}$$

풀이④: 로그 함수의 미분

$$\sin x = x \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

$$x = \pi y$$

$$\sin \pi y = \pi y \prod_{n=1}^{\infty}\left(1-\frac{y^2}{n^2}\right)$$

$$\ln(\sin\pi y) = \ln \pi y + \ln(1-y^2) + \ln\left(1-\frac{y^2}{2^2}\right) + \ln\left(1-\frac{y^2}{3^2}\right) + \cdots$$

$$= \ln \pi y + \sum_{n=1}^{\infty} \ln\left(1-\frac{y^2}{n^2}\right)$$

$$\frac{\pi \cos \pi y}{\sin \pi y} = \frac{\pi}{\pi y} + \sum_{n=1}^{\infty} \frac{\frac{-2y}{n^2}}{1-\frac{y^2}{n^2}}$$

$$= \frac{1}{y} + \sum_{n=1}^{\infty} \frac{-2y}{n^2-y^2}$$

$$-\frac{\pi \cos \pi y}{2y \sin \pi y} = -\frac{1}{2y^2} + \sum_{n=1}^{\infty} \frac{1}{n^2-y^2}$$

$$\frac{1}{2y^2} - \frac{\pi\cos\pi y}{2y\sin\pi y} = \sum_{n=1}^{\infty}\frac{1}{n^2-y^2}$$

$$\lim_{y\to 0}\left(\frac{1}{2y^2} - \frac{\pi\cos\pi y}{2y\sin\pi y}\right) = \lim_{y\to 0}\sum_{n=1}^{\infty}\frac{1}{n^2-y^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\lim_{y\to 0}\left(\frac{1}{2y^2} - \frac{\pi\cos\pi y}{2y\sin\pi y}\right) = \lim_{y\to 0}\frac{\sin\pi y - \pi y\cos\pi y}{2y^2\sin\pi y}$$

$$= \lim_{y\to 0} \frac{\pi\cos\pi y - \pi\cos\pi y + \pi^2 y\sin\pi y}{4y\sin\pi y + 2\pi y^2\cos\pi y}$$

$$= \lim_{y\to 0} \frac{\pi^2 y\sin\pi y}{4y\sin\pi y + 2\pi y^2\cos\pi y}$$

$$= \lim_{y\to 0} \frac{\pi^2 \sin\pi y}{4\sin\pi y + 2\pi y\cos\pi y}$$

$$= \lim_{y\to 0} \frac{\pi^2}{4 + 2\pi y\frac{\cos\pi y}{\sin\pi y}}$$

$$\lim_{y\to 0} 2\pi y\frac{\cos\pi y}{\sin\pi y} = \lim_{y\to 0} 2\cos\pi y \cdot \frac{1}{\underbrace{\frac{\sin\pi y}{\pi y}}_{\to 1}} = 2$$

$$\therefore \frac{\pi^2}{4+2} = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

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