Proof of Irrationality of π

Jonggeun Kim*†

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Abstract

π is known as a familiar irrational number. However, in the high school curriculum and undergraduate courses, we accept the fact that π is irrational without any proof. In this article, we review the three proofs of Ivan Niven, Lambert and Cartwright.

Contents

1 Introduction

In 1761, Johann Heinrich Lambert[1] reported the first proof that the number π is irrational, in other words π cannot be expressed as a fraction $\frac{a}{b}$, where a and b are both non-zero integers. In 1873, Charles Hermite[2,3] found another proof which only uses basic calculus. Three simplifications of Hermite's proof are due to Mary Cartwright, Ivan Niven[4], and Nicolas Bourbaki[5]. Another proof, which is a simplification of Lambert's proof, is due to Miklós Laczkovich[6]. Many of these are *reductio ad absurdum.*

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†Sungkyunkwan University, Department of Natural Science, Suwon

[1] Lambert, Johann Heinrich (1768), "Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques", in Berggren.

[2] Hermite, Charles (1873). "Extrait d'une lettre de Monsieur Ch. Hermite à Monsieur Paul Gordan". Journal für die reine und angewandte Mathematik. 76: 303–311.

[3] Hermite, Charles (1873). "Extrait d'une lettre de Mr. Ch. Hermite à Mr. Carl Borchardt". Journal für die reine und angewandte Mathematik. 76: 342–344.

[4] Niven, Ivan (1947), "A simple proof that π is irrational", Bulletin of the American Mathematical Society, vol. 53, no. 6, p. 509.

[5] Bourbaki, Nicolas (1949), Fonctions d'une variable réelle, chap. I–II–III, Actualités Scientifiques et Industrielles, vol. 1074, Hermann, pp. 137–138.

[6] Laczkovich, Miklós (1997), "On Lambert's proof of the irrationality of π", American Mathematical Monthly, vol. 104, no. 5, pp. 439–443.

2 Ivan Niven's proof

Lemma 2.1. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ *for all* $x \in \mathbb{R}$, $n \in \mathbb{N}$

Proof. Note that Taylor series of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. By the ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0.$$

Therefore for any $x \in \mathbb{R}$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, which means $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any $x \in \mathbb{R}$. □

Suppose that π is rational, such that $\pi = \frac{a}{b}$. ($a, b \in \mathbb{Z}^+$, a,b are rel. prime.)

Definition 2.1. *Let* $f \in C^{2n+2}$ *be*

$$f(x) = \frac{x^n(a-bx)^n}{n!} \tag{1}$$

where $n \in \mathbb{N}$.

Definition 2.2. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be*

$$F(x) = f(x) - f''(x) - \cdots + (-1)^n f^{(2n)}(x) \tag{2}$$

where $n \in \mathbb{N}$.

In this section, f and F are used as defined above.

Lemma 2.2. $f(x) = f(\pi - x)$ *for all* $x \in \mathbb{R}$.

Proof.

$$f(x) = \frac{x^n(a-bx)^n}{n!} = \frac{x^n b^n(\frac{a}{b} - x)^n}{n!} = \frac{(\pi - x)^n(bx)^n}{n!}$$

$$bx = a - a + bx = a - b(\frac{a}{b} - x) = a - b(\pi - x)$$

$$f(x) = \frac{(\pi - x)^n(bx)^n}{n!} = \frac{(\pi - x)^n(a - b(\pi - x))^n}{n!} = f(\pi - x).$$

Therefore, $f(x) = f(\pi - x)$ for all $x \in \mathbb{R}$. □

Also, since $f(x)$ is a polynomial of degree $2n$, $f(x)$ can be expressed as

$$f(x) = \frac{x^n(a-bx)^n}{n!} = \frac{c_0}{n!}x^n + \frac{c_1}{n!}x^{n+1} + \cdots + \frac{c_n}{n!}x^{2n},$$

where $c_n \in \mathbb{Z}$.

Lemma 2.3. $F(0) + F(\pi) \in \mathbb{Z}$

Proof.

Claim 1) $F(0) \in \mathbb{Z}$.

It can be easily seen that

$$F(0) = f(0) - f''(0) - \cdots + (-1)^n f^{(2n)}(0).$$

by Eq. (2). Therefore, if $f^{(k)}(0) \in \mathbb{Z}$ for all $k \in \mathbb{N}$, $F(0) \in \mathbb{Z}$.

Case 1-1) For $k < n$,

$$\begin{aligned} f(x) &= \frac{c_0}{n!}x^n + \frac{c_1}{n!}x^{n+1} + \cdots + \frac{c_n}{n!}x^{2n} \\ f^{(k)}(x) &= C_0 x^{n-k} + C_1 x^{n+1-k} + \cdots + C_n x^{2n-k}, \end{aligned}$$

where $C_i = \frac{c_i}{n!}\prod_{j=1}^{k}(n+i+1-j)$ for $i \in \{0, 1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots, k\}$.

$$\therefore f^{(k)}(0) = C_0 0^{n-k} + C_1 0^{n+1-k} + \cdots + C_n 0^{2n-k} = 0.$$

Therefore, if $k < n$, $f^{(k)}(0) = 0$.

Case 1-2) For $n \le k \le 2n$,

$$\begin{aligned} f(x) &= \frac{c_0}{n!}x^n + \frac{c_1}{n!}x^{n+1} + \cdots + \frac{c_n}{n!}x^{2n} \\ f^{(k)}(x) &= 0 + 0 + \cdots + \frac{c_k}{n!}k! + \frac{c_k}{n!}(k+1)!x + \cdots \\ f^{(k)}(0) &= \frac{c_k}{n!}k! \end{aligned}$$

Therefore, if $n \le k \le 2n$, $f^{(k)}(0) = \frac{c_k}{n!}k! \in \mathbb{Z}$.

Case 1-3) For $k > 2n$,

Since $\deg f(x) = 2n$, it is trivial that $f^{(k)}(x) = 0$. Therefore, if $k > 2n$, $f^{(k)}(x) = 0 \in \mathbb{Z}$.

Summarizing the above discussion, it can be shown that $f^{(k)}(0) \in \mathbb{Z}$ for all $k \in \mathbb{N}$ and

$$f^{(k)}(0) = \begin{cases} 0, & \text{if } k < n, \\ \frac{c_k}{n!}k!, & \text{if } n \le k \le 2n, \\ 0, & \text{if } k > 2n. \end{cases}$$

Claim 2) $F(\pi) \in \mathbb{Z}$.

$$F(\pi) = f(\pi) - f''(\pi) - \cdots + (-1)^n f^{(2n)}(\pi).$$

Therefore, If $f^{(k)}(\pi) \in \mathbb{Z}$ for all $k \in \mathbb{N}$, $F(\pi) \in \mathbb{Z}$.

By lemma 2.2 and the chain rule,

$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x).$$

Substituting π to x in (2), we get

$$f^{(k)}(\pi) = (-1)^k f^{(k)}(0). \tag{3}$$

By Eq. (3) and the Claim 1, we get $f^{(k)}(\pi) \in \mathbb{Z}$ for all $k \in \mathbb{N}$.

From both cases, we prove that

$$F(0) + F(\pi) \in \mathbb{Z}.$$

$\square$

Lemma 2.4. $F(0) + F(\pi) = \int_0^\pi f(x) \sin x \, dx$.

Proof. By Eq. (2),

$$\begin{aligned}
F''(x) &= f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots + (-1)^{(n)} f^{(2n)}(x) \\
F(x) &= f(x) - \{f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots + (-1)^{(2n)}(x)\} \\
&= f(x) - F''(x) \\
f(x) &= F(x) + F''(x). \qquad (4)
\end{aligned}$$

Multiply $\sin x$ on both sides of Eq. (4), we get

$$f(x) \sin x = F(x) \sin x + F''(x) \sin x. \qquad (5)$$

When we integrate both sides of Eq. (5) over $(0, \pi)$, we get the desired identity:

$$\begin{aligned}
&\int_0^\pi f(x) \sin x dx \\
&= \int_0^\pi F(x) \sin x dx + \int_0^\pi F''(x) \sin x dx \\
&= [-F(x) \cos x]_0^\pi + \int_0^\pi F'(x) \cos x dx \\
&+ [F'(x) \sin x]_0^\pi - \int_0^\pi F'(x) \cos x dx \\
&= F(0) + F(\pi).
\end{aligned}$$

$\square$

Theorem 2.5 (Niven's Proof)**.** *π is irrational.*

Proof. With some algebraic manipulations we get the following inequality.

$$\begin{aligned}
(\pi - x)bx &= -b(x^2 - \pi x) \\
&= -b\left(x - \frac{\pi}{2}\right)^2 + \frac{b\pi^2}{4} \\
&= -b\left(x - \frac{\pi}{2}\right)^2 + \frac{a\pi}{4} \leq \frac{a\pi}{4}.
\end{aligned}$$

By the above inequality, we can show that

$$\begin{aligned}
0 < (\pi - x)(bx) &\leq \frac{a\pi}{4}, \\
0 < \frac{(\pi - x)^n (bx)^n}{n!} &\leq \frac{1}{n!}\left(\frac{a\pi}{4}\right)^n, \\
0 < f(x) &\leq \frac{1}{n!}\left(\frac{a\pi}{4}\right)^n. \qquad (6)
\end{aligned}$$

for $x \in (0, \pi)$.

For $x \in (0, \pi)$, $0 < \sin x < 1$. Using this fact with Eq. (6), this leads

$$\begin{aligned} 0 &< f(x) \sin x < \frac{1}{n!} \left(\frac{a\pi}{4} \right)^n, \\ 0 &< \int_0^\pi f(x) \sin x \, dx < \int_0^\pi \frac{1}{n!} \left(\frac{a\pi}{4} \right)^n dx = \frac{\pi}{n!} \left(\frac{a\pi}{4} \right)^n. \end{aligned}$$

Let $a_n = \frac{\pi}{n!} \left(\frac{a\pi}{4} \right)^n$. We can see that

$$\lim_{n \to \infty} a_n = 0$$

by lemma 2.1.

Hence, for sufficiently large n,

$$\frac{\pi}{n!} \left(\frac{a\pi}{4} \right)^n < 1.$$

By two inequalities we stated above, we show that

$$0 < \int_0^\pi f(x) \sin x \, dx < \frac{\pi}{n!} \left(\frac{a\pi}{4} \right)^n < 1. \tag{7}$$

The Eq. (7) contradicts with lemma 2.3 and 2.4 because there is no integer between 0 and 1. Therefore, π is irrational. $\square$

3 Lambert's proof

Definition 3.1. *A $finite$ $continued$ $fraction$ is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

where $a_0 \in \mathbb{R}, a_k \in \mathbb{R}^+$ for $k \in \{1, 2, \ldots, n\}$. The continued fraction is called simple if $a_0, a_1, \ldots, a_n \in \mathbb{Z}$.

Lemma 3.1. *Every irrational number can be written as a infinite continued fraction.*

Proof.

Claim 1) Every $finite$ $simple$ $continued$ $fraction$ can be written as a rational number.

We will prove it using mathematical induction. For $n = 1$, we have

$$[a_0; a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1},$$

which is rational. Now, we assume that for the positive integer k the simple continued fraction $[a_0; a_1, a_2, \ldots, a_k]$ is rational whenever $a_0 \in \mathbb{Z}, a_i \in \mathbb{Z}^+$ for $i \in \{1, 2, \ldots, k\}$. Let $a_0 \in \mathbb{Z}, a_i \in \mathbb{Z}^+$ for $i \in \{1, 2, \ldots, k+1\}$. Note that

$$[a_0; a_1, \ldots, a_{k+1}] = a_0 + \frac{1}{[a_1; a_2, \ldots, a_k, a_{k+1}]}.$$

By the induction hypothesis, $[a_1; a_2, \ldots, a_k, a_{k+1}]$ is rational; hence, there are integers r and s, with $s \neq 0$, such that this continued fraction equals r/s. Then

$$[a_1; a_2, \ldots, a_k, a_{k+1}] = a_0 + \frac{1}{r/s} = \frac{a_0 r + s}{r},$$

which is again a rational number.

Claim 2) Every irrational number can be written as a *infinite continued fraction.*

The contrapositive of Claim 1 is Claim 2. Therefore, every irrational number can be written as a infinite continued fraction. $\square$

Lemma 3.2. $\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}},$

for $x \in \mathbb{R}$.

Proof. Using Maclaurin series, $tanx$ can be expressed as

$$
\begin{aligned}
\tan x = \frac{\sin x}{\cos x} &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} \\
&= \frac{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} = \frac{x}{\frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}} \\
&= \frac{x}{\frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) + \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}} \\
&= \frac{x}{\frac{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) + \left(-\frac{2}{3!}x^2 + \frac{4}{5!}x^4 - \frac{6}{7!}x^6 + \cdots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}} \\
&= \frac{x}{1 - \frac{\left(\frac{2}{3!}x^2 - \frac{4}{5!}x^4 + \frac{6}{7!}x^6 - \cdots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}} = \frac{x}{1 - \frac{x^2\left(\frac{2}{3!} - \frac{4}{5!}x^2 + \frac{6}{7!}x^4 - \cdots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}} \\
&= \frac{x}{1 - \frac{x^2}{\frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}{\left(\frac{2}{3!} - \frac{4}{5!}x^2 + \frac{6}{7!}x^4 - \cdots\right)}}} = \frac{x}{1 - \frac{x^2}{\frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}{\left(\frac{1}{3} - \frac{1}{53}x^2 + \frac{1}{75}x^4 - \cdots\right)}}} \\
&= \frac{x}{1 - \frac{x^2}{\frac{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) - 3\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \cdots\right) + 3\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \cdots\right)}{\left(\frac{1}{3} - \frac{1}{53}x^2 + \frac{1}{75}x^4 - \cdots\right)}}} \\
&= \frac{x}{1 - \frac{x^2}{\frac{3\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \cdots\right) + \left(-\frac{2}{5}\frac{x^2}{3!} + \frac{4}{7}\frac{x^4}{5!} - \cdots\right)}{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \cdots\right)}}} \\
&= \frac{x}{1 - \frac{x^2}{3 - \frac{\left(\frac{2}{5}\frac{x^2}{3!} - \frac{4}{7}\frac{x^4}{5!} + \cdots\right)}{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \cdots\right)}}} = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2\left(\frac{2}{5}\frac{1}{3!} - \frac{4}{7}\frac{x^2}{5!} + \cdots\right)}{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \cdots\right)}}} \\
&= \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{\frac{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \cdots\right)}{\left(\frac{2}{5}\frac{1}{3!} - \frac{4}{7}\frac{x^2}{5!} + \cdots\right)}}}}
\end{aligned}
$$

$\cdots$

Repeating this procedure, we can prove that

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}},$$

for $x \in \mathbb{R}$. $\square$

Lemma 3.3. *If x is rational,* $\tan x$ *is irrational.*

Proof. Take $x = \frac{p}{q}$, such that $p, q \in \mathbb{Z}$, p, q are rel. prime.

$$\tan\left(\frac{p}{q}\right) = \cfrac{\left(\frac{p}{q}\right)}{1 - \cfrac{\left(\frac{p}{q}\right)^2}{3 - \cfrac{\left(\frac{p}{q}\right)^2}{5 - \cfrac{\left(\frac{p}{q}\right)^2}{7 - \ddots}}}}$$

$\therefore \tan(\frac{p}{q})$ is an infinite continued fraction. Therefore, by Theorem 3.1, if x is a rational number, $\tan x$ is irrational. $\square$

Theorem 3.4 (Lambert's Proof)**.** *π is irrational.*

Suppose that π is a rational number. Then, $\frac{\pi}{4}$ is also a rational number. Put $x = \frac{\pi}{4}$ into $\tan x$,

$$\tan\frac{\pi}{4} = 1.$$

However, it contradicts with lemma 3.3. Therefore, π is irrational.

$\square$

4 Cartwright's proof

Define $I_n : \mathbb{R} \to \mathbb{R}$ as $I_n(x) = \int_{-1}^{1}(1-z^2)^n \cos(xz)dz$, where n is a non-negative integer.

Lemma 4.1. *For $n \geq 2$, $x^2 I_n(x) = 2n(2n-1)I_{n-1}(x) - 4n(n-1)I_{n-2}(x)$.*

Proof.

$$
\begin{aligned}
I_n(x) &= \int_{-1}^{1}(1-z^2)^n\cos(xz)dz = \left[\frac{(1-z^2)^n\sin(xz)}{x}\right]_{-1}^{1} + \int_{-1}^{1}\frac{2nz(1-z^2)^{n-1}\sin(xz)}{x}dz \\
&= 2n\int_{-1}^{1}\frac{z(1-z^2)^{n-1}\sin(xz)}{x}dz \\
&= -2n\left[\frac{(1-z^2)^{n-1}\cos(xz)}{x}\right]_{-1}^{1} + 2n\int_{-1}^{1}\frac{\{(1-z^2)^{n-1}-2(n-1)z^2(1-z^2)^{n-2}\}\cos(xz)}{x^2}dz \\
&= \frac{2n}{x^2}\int_{-1}^{1}\{(1-z^2)^{n-1}-2(n-1)z^2(1-z^2)^{n-2}\}\cos(xz)dz \\
&= \frac{2n}{x^2}\int_{-1}^{1}(1-z^2)^{n-1}\cos(xz)dz - \frac{2n(n-1)}{x^2}\int_{-1}^{1}2z^2(1-z^2)^{n-2}\cos(xz)dz.
\end{aligned}
$$

$$
\therefore I_n(x) = \frac{2n}{x^2}I_{n-1}(x) - \frac{4n(n-1)}{x^2}\int_{-1}^{1}z^2(1-z^2)^{n-2}\cos(xz)dz.
$$

On the other hand,

$$
\begin{aligned}
I_{n-1}(x) - I_{n-2}(x) &= \int_{-1}^{1}(1-z^2)^{n-1}\cos(xz)dz - \int_{-1}^{1}(1-z^2)^{n-2}\cos(xz)dz \\
&= \int_{-1}^{1}(1-z^2)^{n-2}(1-z^2-1)\cos(xz)dz \\
&= -\int_{-1}^{1}z^2(1-z^2)^{n-2}\cos(xz)dz.
\end{aligned}
$$

$$
\begin{aligned}
\therefore I_n(x) &= \frac{2n}{x^2}I_{n-1}(x) + \frac{4n(n-1)}{x^2}\{I_{n-1}(x) - I_{n-2}(x)\} \\
&= \frac{2n(2n-1)}{x^2}I_{n-1}(x) - \frac{4n(n-1)}{x^2}I_{n-2}(x).
\end{aligned}
$$

Therefore, For $n \geq 2$, $x^2 I_n(x) = 2n(2n-1)I_{n-1}(x) - 4n(n-1)I_{n-2}(x)$. □

If $J_n(x) = x^{2n+1}I_n(x)$, then this becomes

$$
J_n(x) = 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2 J_{n-2}(x).
$$

for $n \geq 2$. Furthermore,

$$
J_0(x) = 2\sin x, J_1(x) = -4x\cos x + 4\sin x.
$$

Lemma 4.2. *For $n \in \mathbb{Z}^+$, $J_n(x) = n!(P_n(x)\cos x + Q_n(x)\sin x)$, where $P_n(x), Q_n(x)$ are polynomials of degree $\leq n$ with integer coefficients (depending on n).*

Proof. We will prove the statement by mathematical induction. For $n = 0$ and $n = 1$, we already proved them previously. Now, we assume that for the positive integers $1, 2, \ldots, k$,

$$
J_k(x) = k!(P_k(x)\cos x + Q_k(x)\sin x).
$$

Note that

$$J_n(x) = 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2 J_{n-2}(x).$$

$$\begin{aligned}
J_{k+1}(x) &= 2(k+1)(2k+1)J_k(x) - 4k(k+1)x^2 J_{k-1}(x) \\
&= 2(k+1)(2k+1)k!(P_k(x)\cos x + Q_k(x)\sin x) \\
&- 4k(k+1)x^2(k-1)!(P_{k-1}(x)\cos x + Q_{k-1}(x)\sin x) \\
&= (k+1)!\{(2(2k+1)P_k(x) - 4x^2 P_{k-1}(x))\cos x \\
&+ (2(2k+1)Q_k(x) - 4x^2 Q_{k-1}(x))\sin x\}.
\end{aligned}$$

Let $P_{k+1} = 2(2k+1)P_k(x) - 4x^2 P_{k-1}(x)$, $Q_{k+1} = 2(2k+1)Q_k(x) - 4x^2 Q_{k-1}(x)$. By the induction hypothesis, $P_{k+1}(x), Q_{k+1}$ are polynomials of degree $\leq n$, and with integer coefficients. Therefore, by mathematical induction, for all $n \in \mathbb{Z}^+$,

$$J_n(x) = n!(P_n(x)\cos x + Q_n(x)\sin x),$$

where $P_n(x), Q_n(x)$ are polynomials of degree $\leq n$, and with integer coefficients.

$\square$

Theorem 4.3 (Cartwright's Proof)**.** *π is irrational.*

Suppose that π is rational. Then, $\frac{\pi}{2}$ is also rational and can be expressed as $\frac{a}{b}$ such that $a, b \in \mathbb{Z}^+$, a,b are rel. prime. Note that

$$x^{2n+1} I_n(x) = n!(P_n(x)\cos x + Q_n(x)\sin x). \tag{8}$$

In (8), take $x = \frac{\pi}{2}$.

$$\begin{aligned}
\left(\frac{a}{b}\right)^{2n+1} I_n\left(\frac{\pi}{2}\right) &= n! \cdot Q_n\left(\frac{\pi}{2}\right). \\
\frac{a^{2n+1}}{n!} I_n\left(\frac{\pi}{2}\right) &= Q_n\left(\frac{\pi}{2}\right) \cdot b^{2n+1}.
\end{aligned}$$

The right side is an integer, so is the left side. However, $0 < I_n\left(\frac{\pi}{2}\right) \leq 2$ since for $y \in (-1, 1)$,

$$0 < (1-z^2)^n \leq 1, 0 < cos\left(\frac{\pi}{2}y\right) \leq 1.$$

$$\begin{aligned}
&0 < (1-z^2)^n \cos\left(\frac{\pi}{2}y\right) \leq 1 \\
&0 < \int_{-1}^{1} (1-z^2)^n \cos\left(\frac{\pi}{2}y\right) \leq 2 \\
&0 < I_n\left(\frac{\pi}{2}\right) \leq 2.
\end{aligned}$$

On the other hand,

$$\lim_{n\to\infty} \frac{a^{2n+1}}{n!} = 0.$$

Hence, for sufficiently large n,

$$0 < \frac{a^{2n+1} I_n\left(\frac{\pi}{2}\right)}{n!} < 1.$$

That is, we could find an integer between 0 and 1. It contradicts with the assumption that π is rational. Therefore, π is irrational.