KEMPE'S PROOF OF FOUR COLOR PROBLEM

HOJUN LEE

ABSTRACT. Francis Guthrie claimed in 1852 the four color problem. We proof two essential lemmas and then solve six color problem. We expand the proof of six color problem into five, four color problem. Kempe published this proof in 1879. However the flaw was discovered in 1890 by Heawood. Although flawed, Kempe's idea was used as one of a basic tool.

1. Basic concept

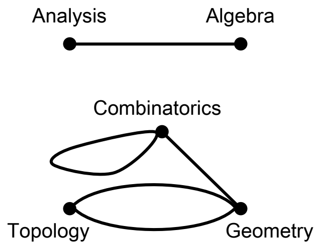
The four color problem was proposed in 1852 by Francis Guthrie. The intuitive statement is "given any separation of a plane into contiguous regions, the regions can be colored using at most four colors so that no two adjacent regions have the same color". However this statement is not mathematical. We need to construct some mathematical concepts and substitute the ordinary concepts, such as separation, contiguous regions, etc., with the mathematical concepts.
To understand the statement of four color problem, we introduce some essential concepts such as planar graph, degree, n-colorable, etc.

Definition

A **graph** G is composed of two types of objects, a vertex set V and an edge set E such that each edge $e \in E$ is associated with an unordered pair of vertices, denoted as $G = (V, E)$.

Example 1

Let $V = \{Analysis, Algebra, Topology, Geometry, Combinatorics\}$
and $E = \{(Analysis, Algebra), (Topology, Geometry), (Topology, Geometry),$
$(Geometry, Combinatorics), (Combinatorics, Combinatorics)\}$.
Then the graph $G = (V, E)$ is as follows:

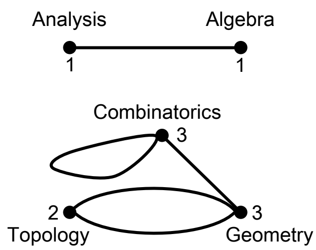


Definition

(1) A **loop** is an edge incident on a single vertex.
(2) If vertex $v \in V$ is shared by k edges, k is called the **degree** of vertex v, and denoted by $d(v) = k$. If the vertex v has a loop, then the loop contributes 2 to the degree of v.
(3) A vertex v is **adjacent** to another vertex w if there is an edge (v, w)
(4) **Multiple edges**(or parallel edges) are two or more edges that are incident to the same two vertices.
(5) A **planar graph** is a graph that may be embedded in the plane without intersecting edges.
(6) A graph G is said to be n-**colorable**, denoted by $c(G) = n$, if it's possible to assign one of n colors to each vertex in such a way that no two connected vertices have the same color.

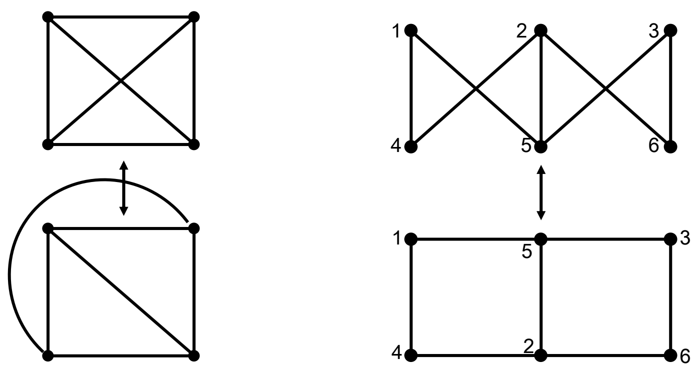
Example 2

Let G define as above example 1.

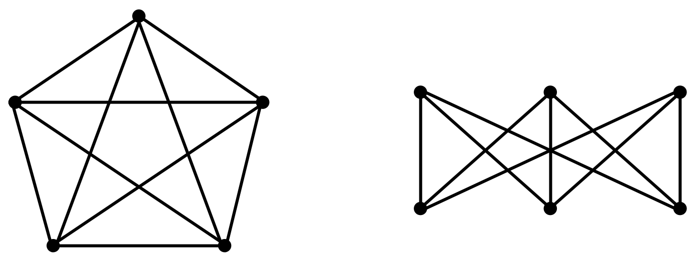


Then $(Topology, Geometry)$ and $(Topology, Geometry)$ are mutiple edges. And there is one loop, $(Combinatorics, Combinatorics)$. The numbers near by each of vertices are degree of them. It is clear that the Graph 2 is a planar graph.

Example 3



Isomorphic planar graphs



K_5 $K_{3,3}$

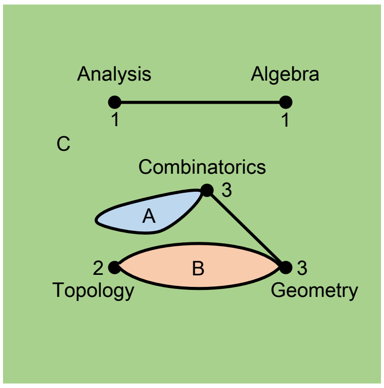
Non-planar graphs

Definition

The **faces** of a planar graph are the areas which are surrounded by edges

Example 4

Let G define as above example 1.



The faces of Graph G are A and B surrounded by $\{(Combinatorics, Combinatorics)\}$ and $\{(Topology, Geometry), (Topology, Geometry)\}$ respectively.
The important thing is that C is a face of Graph G either. The face C is surrounded by all edges of graph G.

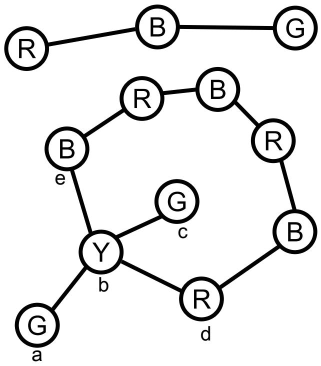
Remark

$G(i, j)$ is a subgraph of G consisting of the vertices that are colored with colors i and j only, and edges connecting two of them $G^c(i, j, v)$ is the connected component of $G(i, j)$ containing vertex v.

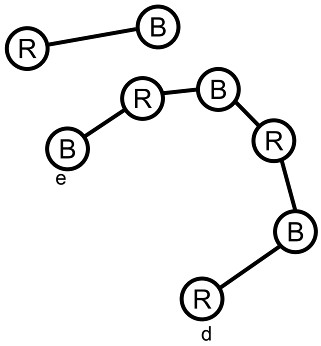
Definition

(1) A path in $G(i, j)$, called a **Kempe chain** and denoted by $Ch(i, j, u, v)$, joining vertices u and v, that is a sequence of edges and vertices painted only with colors i and j.
(2) When vertex $w \notin Ch(i, j, u, v)$, $Ch(i, j, u, v)$ together with w as well as its two edges connected to u and v forms a **Kempe circle**, which is denoted by $Ch(i, j, u, v) + w$.

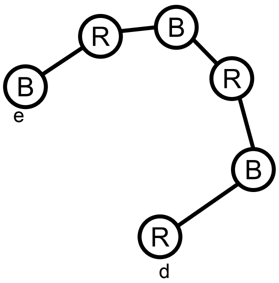
Example 5



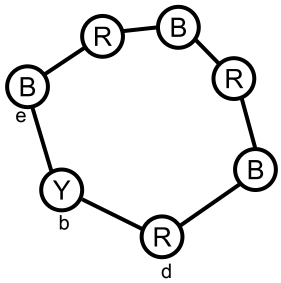
G



$G(R, B)$



$G^c(R, B, d)(= Ch(R, B, d, e))$



$Ch(R, B, d, e) + b$

2. Set-up

In order to deal with maps mathematically, we consider each of territories as a vertex. We also consider the adjacent relation between two territories as an edge, which is associated with corresponding vertices. For elegant proof, we now think of graph as connected and loopless and non-multiple edges graph. We can easily know the reason why it is connected, loopless, and non-multiple edges by imagining how maps look like.
Now, we introduce famous facts known as **Euler's formula** and **Euler's Theorem**.

Euler's formular in a planar graph

Let G be a finite, connected, and planar graph and let $n(v)$, $n(e)$, and $n(f)$ be respectively the number of vertices, edges, and faces. Then

$$n(v) - n(e) + n(f) = 2$$

Euler's Theorem

Let $n(d_i)$ be the number of vetices of degree d_i and D is the maximum degree. Then

$$2 \times n(e) = \sum_{i=1}^{D} i \times n(d_i)$$

Intuitively, the left-hand side means two times the number of edges, and the right-hand side means the sum of degrees of all vertices. It is probably clear since each edge has two vertices.

3. Lemmas

Lemma 1

Let G be a planar graph. If $n(v) \geq 3$ then

$$n(e) \leq 3 \times n(v) - 6,$$

where $n(v)$ and $n(e)$ are the number of vertices and edges respectively.

Proof

Since each edge is shared by two faces and each face is bounded by three edges at least,

$$2 \times n(e) \geq 3 \times n(f).$$

Using Euler's formula,

$$2 = n(v) - n(e) + n(f) \leq n(v) - n(e) + \frac{2}{3} \times n(e) = n(v) - \frac{1}{3} \times n(e)$$
$$\therefore n(e) \leq 3 \times n(v) - 6 \quad \square$$

Lemma 2

Let G_n be a planar graph with $n\,(n \geq 6)$ vertices. WLOG the vertex v_n has the smallest degree. Then $d(v_n) \leq 5$ and $G_{n-1} = G_n - v_n$ is also a planar graph.

Proof

$G_{n-1} = G_n - v_n$ is trivially planar. And

$$6 \times n(v) - 2 \times n(e) = 6 \times \sum_{i=1}^{D} n(d_i) - \sum_{i=1}^{D} i \times n(d_i) = \sum_{i=1}^{D} (6-i) \times n(d_i) \geq 12.$$

We use the Euler's theorem in the first equality and lemma 1 in the inequality. Since $12 \geq 0$ and $6 - i \leq 0$ for all $i \geq 6$, there is at least one vertex of degree 5 or less in the planar graph G_n $\square$

4. Six Color Theorem

Theorem 1(The Six Color Theorem)

Every planar graph is six-colorable.

Proof

For any planar graph G_n with n vertices, we show that G_n is six-colorable for all n, using mathematical induction.

(i) When $n \leq 6$, it is trivial that G_n is six-colorable.
(ii) Assume that G_k is six-colorable where $k \geq 6$.
(iii) Think about G_{k+1}. Since $k+1 \geq 6$, by lemma 2, there is a vertex (wlog v_{k+1}) of which degree is 5 or less ($i.e.\, d(v_{k+1}) \leq 5$). Using lemma 2 again, $G' := G_{k+1} - v_{k+1}$ is also a planar graph. Since G' has k vertices, by hypothesis, G' is six-colorable. Since $d(v_{k+1}) \leq 5$, we can paint the vertex v_{k+1} the rest color. $\square$

5. Five Color Theorem

Theorem 2(The Five Color Theorem)

Every planar graph is five-colorable.

Proof

Use mathematical induction.

(i) When $n \leq 5$, it is trivial that G_n is five-colorable.
(ii) Assume that G_k is five-colorable where $k \geq 5$.
(iii) Think about G_{k+1}. Since $k+1 \geq 6$, by lemma 2, there is a vertex (wlog v_{k+1}) of which degree is 5 or less ($i.e.\, d(v_{k+1}) \leq 5$). Using lemma 2 again, $G' := G_{k+1} - v_{k+1}$ is also a planar graph. Since G' has k vertices, by hypothesis, G' is five-colorable.
Let $C(v) = \{color(u) | color(u)$ is the color of vertex u in $G - v$, and u is adjacent to v in $G\}$. There are two cases(Case 1, Case 2) to discuss.

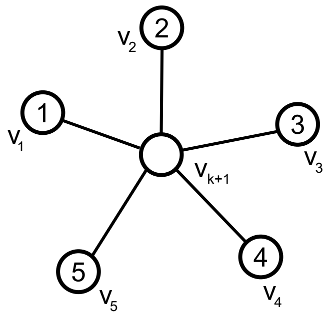
Case 1$(|C(v_{k+1})| \leq 4)$

We can paint the vertex v_{k+1} the rest color.

Case 2$(|C(v_{k+1})| = 5)$

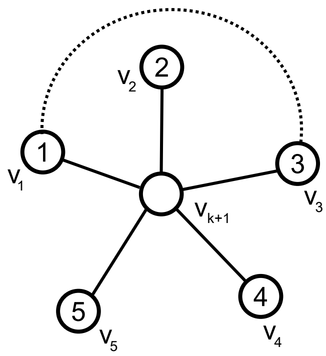
Clearly, $d(v_{k+1}) = 5$. Let v_1, v_2, v_3, v_4, v_5 be adjacent to v_{k+1} in clockwise order. Then there are two cases(Case 2.1 and Case 2.2) to discuss.

Case 2.1$(v_1 \notin G^c_{k+1}(color(v_1), color(v_3), v_3))$



We can reverse the coloration on $G^c_{k+1}(color(v_1)$, $color(v_3)$, $v_3)$ and assign color number 3 to v_{k+1}.

Case 2.2$(v_1 \in G^c_{k+1}(color(v_1), color(v_3), v_3))$



Since G_{k+1} is a planar graph, a Kempe Circle $Ch(color(v_1), color(v_3), v_1, v_3)$ + v_{k+1} seperates $G^c_{k+1}(color(v_2), color(v_4), v_2)$ of $G_{k+1}(color(2), color(4))$ from $G^c_{k+1}(color(2), color(4), v_4)$. We can reverse the coloration on $G^c_{k+1}(color(v_2)$, $color(v_4)$,$v_2)$ and assign color number 2 to v_{k+1}. $\square$

6. Four Color Theorem

Theorem 3(The Four Color Theorem)

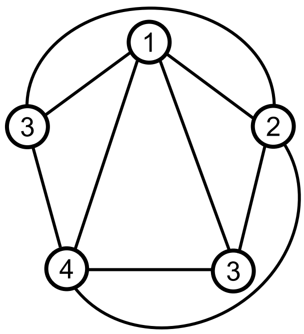
Every planar graph is four-colorable.

Kempe's Proof

Use mathematical induction.

(i) When $n \leq 4$, it is trivial that G_n is four-colorable.

(ii) When $n = 5$, the maximal planar graph with 5 vertices is the full graph deleting an edge, i.e., the planar graph with 5 vertices and 9 edges, which is denoted by G_5^*. Any G_5 is a subgraph of G_5^*, and $c(G_5) \leq c(G_5^*)$. Since $c(G_5^*) = 4$, $c(G_5) = 4$.



(iii) Assume that G_k is four-colorable where $k \geq 5$.

(iv) Think about G_{k+1}. Since $k+1 \geq 6$, by lemma 2, there is a vertex (wlog v_{k+1}) of which degree is 5 or less ($i.e.\, d(v_{k+1}) \leq 5$). Using lemma 2 again, $G' := G_{k+1} - v_{k+1}$ is also a planar graph. Since G' has k vertices, by hypothesis, G' is four-colorable.
Note that $|C(v_{k+1})| \leq 4$ and $|C(v_{k+1})| \leq d(v_{k+1}) \leq 5$. There are 3 cases (Case 1 - Case 3) to discuss.

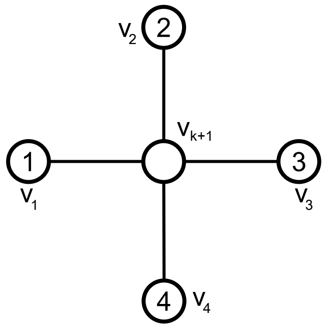
Case 1($|C(v_{k+1})| \leq 3$)

We can paint the vertex v_{k+1} the rest color.

Case 2($|C(v_{k+1})| = 4$ and $d(v_{k+1}) = 4$)

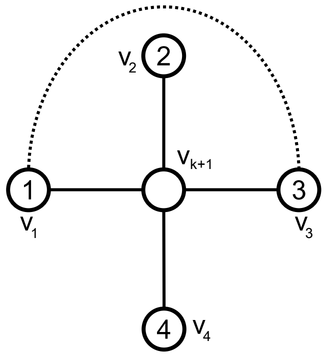
Let v_1, v_2, v_3, v_4 be adjacent to v_{k+1} in clockwise order. There are two cases(Case 2.1, Case 2.2) to discuss.

Case 2.1($v_1 \notin G^c_{k+1}(color(v_1), color(v_3), v_3)$)



We can reverse the coloration on $G^c_{k+1}(color(v_1),\ color(v_3),\ v_3)$ and assign color number 3 to v_{k+1}.

Case 2.2($v_1 \in G^c_{k+1}(color(v_1), color(v_3), v_3)$)



Since G_{k+1} is a planar graph, a Kempe Circle $Ch(color(v_1), color(v_3), v_1, v_3)$ $+\ v_{k+1}$ seperates $G^c_{k+1}(color(v_2), color(v_4), v_2)$ of $G_{k+1}(color(2), color(4))$ from $G^c_{k+1}(color(2), color(4), v_4)$. We can reverse the coloration on $G^c_{k+1}(color(v_2),$ $color(v_4)$,$v_2)$ and assign color number 2 to v_{k+1}.

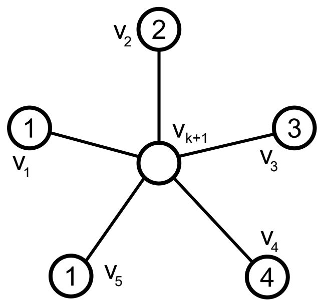
Case 3($|C(v_{k+1})| = 4$ and $d(v_{k+1}) = 5$)

Let v_1, v_2, v_3, v_4, v_5 be adjacent to v_{k+1} in clockwise order. Only two of the five vertices are painted with the same color, and the two vertices with the same color are neighbor or isolate in cyclic order. Therefore, there are 2 cases (Case 3.1 and Case 3.2) to discuss.

Case 3.1

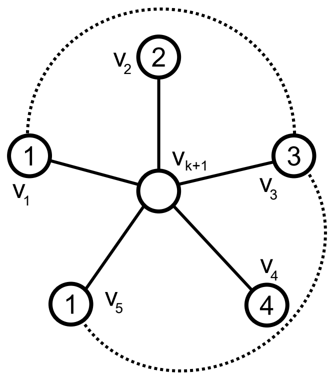
The two vertices with the same color are neighbor in cyclic oreder, wlog, let $color(v_j) = j$ for $1 \leq j \leq 4$ and $color(v_5) = 1$. Consider the subgraph $G_{k+1}(1,3)$ of G_{k+1}, there are 2 cases (Case 3.1.1 and Case 3.1.2) to discuss.

Case 3.1.1($v_3 \notin G^c_{k+1}(1,3,v_1)$ and $v_3 \notin G^c_{k+1}(1,3,v_5)$)



we can reverse the coloration on $G^c_{k+1}(1,3,v_3)$, thus assign color number 3 to v_{k+1}.

Case 3.1.2($v_3 \in G^c_{k+1}(1,3,v_1)$ or $v_3 \in G^c_{k+1}(1,3,v_5)$)

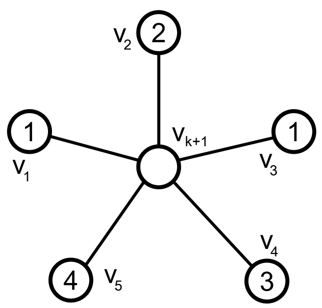


We can find a Kempe chain $Ch(1,2,v_1,v_3)$ or $Ch(1,3,v_5,v_3)$ in $G_{k+1}(1,3)$. Kempe Circle $Ch(1,2,v_1,v_3) + v_{k+1}$ or $Ch(1,3,v_5,v_3) + v_{k+1}$ separates $G^c_{k+1}(2,4,v_2)$ of $G_{k+1}(2,4)$ from $G^c_{k+1}(2,4,v_4)$, we can reverse the coloration on $G^c_{k+1}(2,4,v_2)$ and assign color number 2 to v_{k+1}.

Case 3.2

The two vertices with the same color are isolate in cyclic oreder, wlog, let $c(v_1)$=$c(v_3)$=1, $c(v_2)$=2, $c(v_4)$=3, and $c(v_5)$=4. Consider the subgraph $G_{k+1}(2,4)$ of G_{k+1}, there are 2 cases (Case 3.2.1 and Case 3.2.2) to discuss.

Case 3.2.1($v_2 \notin G^c_{k+1}(2,4,v_5)$)

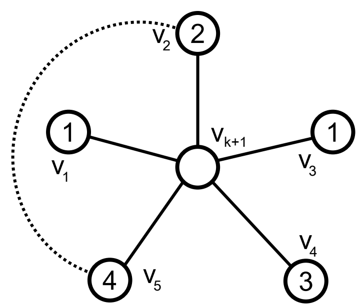


We can reverse the coloration on $G^c_{k+1}(2,4,v_5)$ and assign color number 4 to v_{k+1}.

Case 3.2.2($v_2 \in G^c_{k+1}(2,4,v_5)$)

There are 2 cases (Case 3.2.2.1 and Case 3.2.2.2) to discuss.

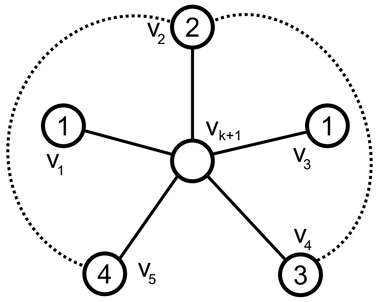
Case 3.2.2.1($v_2 \notin G^c_{k+1}(2,3,v_4)$)



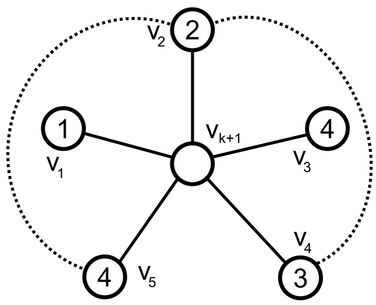
We can reverse the coloration on $G^c_{k+1}(2,3,v_4)$ and assign color number 3 to v_{k+1}.

Case 3.2.2.2($v_2 \in G^c_{k+1}(2,3,v_4)$)

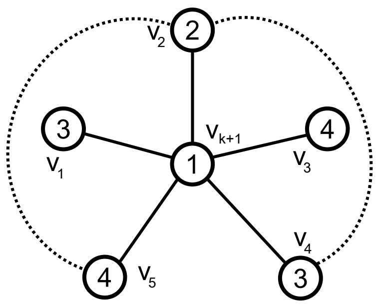
We can find a Kempe chain $Ch(2,3,v_2,v_4)$ in $G_{k+1}(2,3)$.



Kempe circle $Ch(2,3,v_2,v_4) + v_{k+1}$ separates $G^c_{k+1}(1,4,v_3)$ of $G_{k+1}(1,4)$ from $G^c_{k+1}(1,4,v_5)$, we can reverse the coloration on $G^c_{k+1}(1,4,v_3)$.

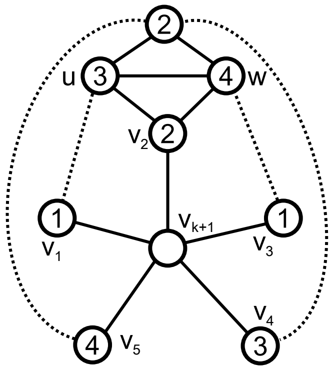


Then we can find a Kempe circle $Ch(2,4,v_2,v_5)$ in $G_{k+1}(2,4)$. Kempe circle $Ch(2,4,v_2,v_5) + v_{k+1}$ separates $G^c_{k+1}(1,3,v_1)$ of $G_{k+1}(1,3)$ from $G^c_{k+1}(1,3,v_4)$, we can reverse the coloration on $G_{k+1}(1,3,v_1)$ and assign color number 1 to v_{k+1}



The Flaw in Kempe's Proof

$\exists\ u \in Ch(2, 3, v_2, v_4)$ and $w \in Ch(2, 4, v_2, v_5)$ $s.t.$ an edge $(u, w) \in E$, where E is an edge set of the graph G.

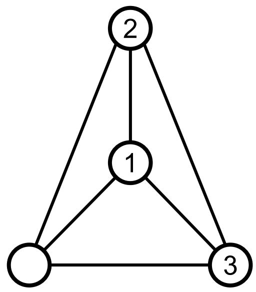


If we were to reverse the color as with Case 3.2.2.2, then u and w are adjacent despite same color.

7. Three Color Problem

Unfortunately, it is impossible to complete painting map using only three colors. The following is a counterexample which disproves the three color problem.

Counterexample



8. Conclusion

This proof is not complete. But the first proof of four color theorem is based on kempe's idea. In order to follow the formal proof, we need to understand Kempe's approach.
Following the principle of mathematical induction in proof, we can make a painting map algorithm using only four colors. This algorithm show that painting a vertex of the largest degree is prioritized. In fact, this process is usual way to paint a map. It means this algorithm is not much different from our intuition, although the flaw exists.

Hojun Lee
Department of Mathematics
Sungkyunkwan University
Suwon, Korea
Email address: hojoon1101@g.skku.edu