Gamma and Zeta Function

주현욱

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I. Gamma Function

1. Definition of Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

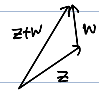
2. Domain of Gamma Function

First, let's prove "$|\Gamma(z)| \le \Gamma(\mathrm{Re}(z))$" $\quad z \in \mathbb{C}$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt = \lim_{c\to\infty} \int_0^c t^{z-1} e^{-t}\, dt = \lim_{\substack{n\to\infty \\ c\to\infty}} \sum_{i=1}^{n} s_i^{z-1} e^{-s_i} \Delta t_i \text{ (Riemann Sum)}$$

$t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = c \quad \Delta t_i = t_i - t_{i-1} \quad s \in [t_{i-1}, t_i]$: Sample point

$$|\Gamma(z)| = \left| \lim_{\substack{n\to\infty \\ c\to\infty}} \sum_{i=1}^{n} s_i^{z-1} e^{-s_i} \Delta t_i \right| \le \lim_{\substack{n\to\infty \\ c\to\infty}} \sum_{i=1}^{n} \left| s_i^{z-1} e^{-s_i} \Delta t_i \right|$$

(

$|z+w| \le |z| + |w|$ $(z, w \in \mathbb{C})$ $\Rightarrow \left|\sum z_i\right| \le \sum |z_i|$)

$$= \lim_{\substack{n\to\infty \\ c\to\infty}} \sum_{i=1}^{n} |s_i^{z}| \, |s_i^{-1} e^{-s_i} \Delta t_i|$$

Let $z = a+bi \quad a, b \in \mathbb{R} \quad |s_i^z| = |s_i^{a+bi}| = |s_i^a| |s_i^{bi}| = |s_i^a| |e^{bi \ln s_i}| = |s_i^a| = |s_i^{\mathrm{Re}(z)}| = s_i^{\mathrm{Re}(z)}$

$$= \lim_{\substack{n\to\infty \\ C\to\infty}} \sum_{i=1}^{n} \left|s_i^{Re(z)}\right| \left|s_i^{-1} e^{-s_i} \Delta t_i\right| = \lim_{\substack{n\to\infty \\ C\to\infty}} \sum_{i=1}^{n} s_i^{Re(z)-1} e^{-s_i} \Delta t_i$$

$$= \lim_{C\to\infty} \int_0^C t^{Re(z)-1} e^{-t} dt = \int_0^\infty t^{Re(z)-1} e^{-t} dt = \Gamma(Re(z)).$$

$$\boxed{|\Gamma(z)| \le \Gamma(Re(z)) \quad (Re(z) > 0)}$$

Next, let's prove "For $\forall x \in (0,\infty)$, $\Gamma(x)$ is absolutely convergent."

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt = \int_0^N t^{x-1} e^{-t} dt + \int_N^\infty t^{x-1} e^{-t} dt \quad (N \in \mathbb{N})$$

i) Let $I_1 = \int_0^N t^{x-1} e^{-t} dt$

Since $t \in [0,N] \rightarrow e^{-t} \le 1$, $t^{x-1} e^{-t} \le t^{x-1}$

$$\Longrightarrow \quad 0 \le \int_0^N t^{x-1} e^{-t} dt \le \int_0^N t^{x-1} dt.$$

$$\int_0^N t^{x-1} dt = \left. \frac{t^x}{x} \right|_{t=0}^{t=N} = \frac{N^x}{x}$$

$$\Rightarrow \quad 0 \le \int_0^N t^{x-1} e^{-t} dt \le \frac{N^x}{x}$$

$\therefore I_1$ is absolutely convergent.

ii) Let $I_2 = \int_N^\infty t^{x-1} e^{-t} dt$

Consider $\lim_{t\to\infty} \frac{t^{x-1}}{e^{\frac{1}{2}t}}$

① $x \in (0,1]$ $\quad 1-x \ge 0$. $\lim_{t\to\infty} \frac{t^{x-1}}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{1}{t^{1-x} e^{\frac{1}{2}t}} = 0 \quad \left(\frac{1}{\infty \times \infty}\right)$

② $x\in(1,\infty)$

$$\lim_{t\to\infty}\frac{t^{x-1}}{e^{\frac{1}{2}t}}=\frac{\infty}{\infty}$$

$$0\le\lim_{t\to\infty}\frac{t^{x-1}}{e^{\frac{1}{2}t}}=\lim_{u\to\infty}\frac{(2u)^{x-1}}{e^{u}}=2^{x-1}\lim_{u\to\infty}\frac{u^{x-1}}{\sum_{k=0}^{\infty}\frac{u^k}{k!}}\le 2^{x-1}\lim_{u\to\infty}\frac{u^{x-1}}{\frac{u^L}{L!}}$$

$(2u=t)$ $\quad e^{u}=e^{\frac{1}{2}t}>1$ $\quad \left(\because \sum_{k=0}^{\infty}\frac{u^k}{k!}\ge\frac{u^L}{L!},\ L\in\mathbb{N}\right)$ $L>x$

$$0\le\lim_{t\to\infty}\frac{t^{x-1}}{e^{\frac{1}{2}t}}\le 2^{x-1}L!\lim_{u\to\infty}\frac{1}{u^{L+1-x}}=0 \quad (\because L+1-x>0)$$

$$\Rightarrow \lim_{t\to\infty}\frac{t^{x-1}}{e^{\frac{1}{2}t}}=0$$

$\therefore$ For $\forall x\in(0,\infty)$, $\displaystyle\lim_{t\to\infty}\frac{t^{x-1}}{e^{\frac{1}{2}t}}=0$

$\Rightarrow$ For $\forall\varepsilon>0$, there exists some M $(M>0)$ such that

for $\forall t\ge M$ $\quad \left|\frac{t^{x-1}}{e^{\frac{1}{2}t}}-0\right|<\varepsilon.$

Let $\varepsilon=1$, $M=N$. Then, for $\forall t\ge N$, $\left|\frac{t^{x-1}}{e^{\frac{1}{2}t}}\right|<1$

$\Rightarrow$ For $\forall t\ge N$, $t^{x-1}<e^{\frac{1}{2}t}$, $t^{x-1}e^{-t}<e^{-\frac{1}{2}t}$

$$\int_N^{\infty}e^{-\frac{1}{2}t}dt=-2e^{-\frac{1}{2}t}\Big|_N^{\infty}=0+2e^{-\frac{1}{2}N}=2e^{-\frac{1}{2}N}$$

$$0<\int_N^{\infty}t^{x-1}e^{-t}dt<\int_N^{\infty}e^{-\frac{1}{2}t}dt=2e^{-\frac{1}{2}N}$$

$\therefore I_2$ is absolutely convergent.)

By i), ii), $\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt$ is absolutely convergent for $x\in(0,\infty)$

$|\Gamma(z)| \leq \Gamma(Re(z))$

$\Rightarrow$ If $Re(z) > 0$, then $|\Gamma(z)|$ is absolutely convergent.

$\boxed{\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \text{ is absolutely convergent if } Re(z) > 0}$

* What if $Re(z) \leq 0$?

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \qquad \text{Let } z = x + iy \quad (x, y \in \mathbb{R}, \underset{\substack{\uparrow \\ Re(z)}}{x} \leq 0)$$

$$t^z = e^{z \ln t} = e^{(x+iy)\ln t} = e^{x \ln t} e^{iy \ln t} = t^x (\cos(y \ln t) + i \sin(y \ln t))$$

$$\Gamma(z) = \int_0^{\infty} t^{x-1} (\cos(y \ln t) + i \sin(y \ln t)) e^{-t} dt$$

$$= \int_0^{\infty} t^{x-1} e^{-t} \cos(y \ln t) dt + i \int_0^{\infty} t^{x-1} e^{-t} \sin(y \ln t) dt.$$

$$Re(\Gamma(z)) = \int_0^{\infty} t^{x-1} e^{-t} \cos(y \ln t) dt = \int_0^{\infty} t^{x-1} e^{-t} \sum_{k=0}^{\infty} \frac{(-1)^k (y \ln t)^{2k}}{(2k)!} dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} \int_0^{\infty} t^{x-1} e^{-t} (\ln t)^{2k} dt = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} \int_0^{\infty} f(t) dt.$$

$$f(t) = t^{x-1} e^{-t} (\ln t)^{2k}$$

If $0 < t \leq \frac{1}{e} \rightarrow \ln t \leq -1 \quad (\ln t)^{2k} \geq 1 \quad f(t) = t^{x-1} e^{-t} (\ln t)^{2k} \geq t^{x-1} e^{-t}$

$$\int_0^{\infty} f(t) dt \geq \int_0^{1/e} f(t) dt \geq \int_0^{1/e} t^{x-1} e^{-t} dt \geq \int_0^{1/e} t^{-1} e^{-t} dt \qquad \begin{matrix} x \in (-\infty, 0] \\ (t^x \geq 1) \end{matrix}$$

$$> \int_0^{1/e} t^{-1} e^{-1} dt = \frac{1}{e} \ln t \Big|_0^{1/e} = -\frac{1}{e} - \frac{1}{e}(-\infty) = \infty$$

$$\mathrm{Re}(\Gamma(z)) = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} \int_0^{\infty} f(t)\,dt = \infty$$

$\Rightarrow$ $\Gamma(z)$ does not converge at $\mathrm{Re}(z) \le 0$ $\boxed{\text{J.H.W}}$

But we can define $\Gamma(z)$ if $\mathrm{Re}(z) < 0$ (except $-1, -2, -3, \cdots$) by analytic continuation using functional equation.

(#) $\Gamma(x+1) = x\Gamma(x)$ $(x > 0)$

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t}\,dt = t^x(-e^{-t})\Big|_0^{\infty} - \int_0^{\infty} x t^{x-1}(-e^{-t})\,dt$$
$$= -\lim_{b\to\infty} b^x e^{-b} + x\int_0^{\infty} t^{x-1} e^{-t}\,dt$$

$$\lim_{b\to\infty} b^x e^{-b} = \lim_{b\to\infty} \frac{b^x}{e^b} = \lim_{b\to\infty} \frac{b^x}{\sum_{k=0}^{\infty} \frac{b^k}{k!}} \le \lim_{b\to\infty} \frac{b^x}{\frac{b^N}{N!}} = N! \lim_{b\to\infty} \frac{1}{b^{N-x}} = 0$$

$\hookrightarrow \left(\sum_{k=0}^{\infty} \frac{b^k}{k!} \ge \frac{b^N}{N!} \quad N \in \mathbb{N},\ N > x \right)$

$$\Rightarrow \Gamma(x+1) = 0 + x\Gamma(x) = x\Gamma(x)$$

By allowing to use the functional equation if $x = z$ $(z \in \mathbb{C})$

We can define $\Gamma(z)$ at $\mathrm{Re}(z) < 0$. ($\Leftarrow$ Analytic Continuation)

$$\Gamma(z)\begin{pmatrix}\text{Applying Analytic}\\ \text{Continuation}\end{pmatrix} = \begin{cases} \int_0^{\infty} t^{z-1} e^{-t}\,dt & (\mathrm{Re}(z) > 0) \\ \Gamma(z+1) = z\Gamma(z) & (z \in \mathbb{C}) \end{cases}$$

$$\lim_{x\to 0} \Gamma(x) = \lim_{x\to 0} \frac{\Gamma(x+1)}{x} = \infty \quad \left(\because \Gamma(1) = \int_0^{\infty} e^{-t}\,dt = -e^{-t}\Big|_0^{\infty} = -(0-1) = 1\right)$$

$\Gamma(-1) = -1 \cdot \Gamma(0)$, $\Gamma(-2) = -2\Gamma(-1)$, $\Gamma(-3) = -3\Gamma(-2)$, $\cdots$

Since $\Gamma(0)$ cannot be defined, we can't also defined $\Gamma(-1), \Gamma(-2), \Gamma(-3), \cdots$

$0, -1, -2, -3, -4, \cdots$: simple pole of Gamma function.

$\therefore$ Domain of $\Gamma = \{ z \mid z \in \mathbb{C} \setminus \{0, -1, -2, \cdots\} \}$

* Analytic continuation

$$f(x) = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x}$$

If $|x| < 1$, then $f(x) = \frac{1}{1-x}$.

Domain of $f(x) = \{ x \mid |x| < 1 \}$

$g(x) = \frac{1}{1-x}$ Domain of $g(x) = \{ x \mid x \neq 1 \}$

$$\text{Analytic continuation of } f(x) = \begin{cases} f(x) = 1 + x + x^2 + \cdots & (|x| < 1) \\ g(x) = \frac{1}{1-x} & (x > 1 \text{ or } x \leq -1) \end{cases}$$

$$f(-2) = 1 - 2 + 4 - 8 + \cdots = \frac{1}{1-(-2)} = \frac{1}{3} \quad (\times)$$

$$\text{Analytic continuation of } f(x) \Big|_{x=-2} = g(-2) = \frac{1}{3} \quad (\circ)$$

The analytic continuation function is unique by $\boxed{\text{identity theorem}}$.
Thus, we can define $\Gamma(z)$ at $\mathrm{Re}(z) < 0$ with any functional equation

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \cdots \text{ etc}$$

3. Defining Factorial of Real Number by Gamma Function.

$\Gamma(x+1) = x\Gamma(x)$ If $x = n \in \mathbb{N}$

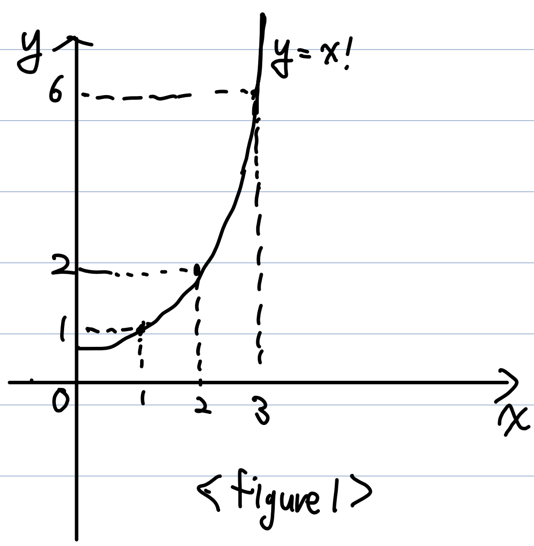
$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot \Gamma(1)$$

$$\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}dt = -e^{-t}\Big|_0^\infty = -(0-1) = 1$$

$\therefore \Gamma(n+1) = n!$ $\Gamma(n) = (n-1)!$

Thus, we can define factorial of real number like $(\frac{1}{2})! = \Gamma(\frac{3}{2})$.

But can we say Gamma is the only function that satisfies $f(x) = (x-1)!$?
(ex) $f(x) = \Gamma(x)\cos^2\pi x \Rightarrow f(n) = \Gamma(n)\cos^2\pi n = \Gamma(n) = (n-1)!$)



<figure 1>

Thus, we need to consider about natural extension of factorial of real number. First, $f(x)$ should be satisfy $f(x+1) = xf(x)$, $f(1) = 1$. Because these are similar with definition of factorial. In addition, the graph of $f(x)$ should be concaved downward like figure 1.

The Bohr-Mollerup Theorem says Gamma function is a unique function satisfying $f(1) = 1$, $f(x+1) = xf(x)$, "Concaved downward".

*Bohr-Mollerup Theurem

If f that defined at $(0,\infty)$ satisfies three properties below.

i) $f(1)=1$
ii) $f(x+1)=xf(x)$ $\quad (x\in(0,\infty))$
iii) $f(x)$ is a log-convex function for $x\in(0,\infty)$

then f is unique.

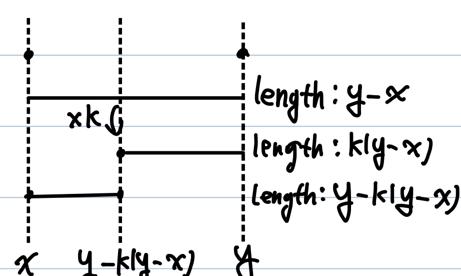
(#) log-convexed function.

$\left(\frac{d^2}{dx^2}\log f(x)>0\right)$

For given function f, if $\log f(x)$ is a log-convex at section I, we say f is a log-convex function at I.
Thus, for $x,y\in I$, $0\le k\le 1$, f satisfies
$\log\{f(kx+(1-k)y)\}\le k\log f(x)+(1-k)\log f(y)$.

(##) Why $h(kx+(1-k)y)\le kh(x)+(1-k)h(y)$ means $h(x)$ is a convex function? $(\log f(x)=h(x))$

$$kx+(1-k)y=kx-ky+y$$
$$=y-k(y-x)$$

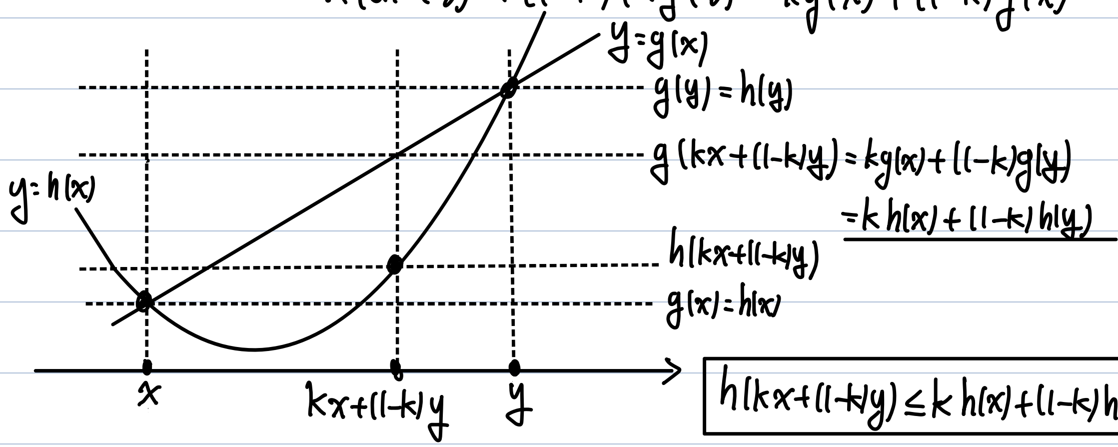


$kx+(1-k)y$ means any points in section (x,y).

$kh(x)+(1-k)h(y)$ also means any points in section $[h(x), h(y)]$ by the same way.

Let $g(x)=ax+b$

$$g(kx+(1-k)y) = a\{kx+(1-k)y\}+b = kax+a(1-k)y+b$$
$$= kax+kb-kb+a(1-k)y+b = kax+kb+a(1-k)y+(1-k)b$$
$$= k(ax+b)+(1-k)(ay+b) = kg(x)+(1-k)g(x)$$



$$\boxed{h(kx+(1-k)y) \le kh(x)+(1-k)h(y)}$$

$h(x) \Rightarrow$ convex function $\Leftrightarrow h(kx+(1-k)y) \le kh(x)+(1-k)h(y)$ $\boxed{\text{J.H.W}}$

Let's prove "Γ is a log-convex function".

$$\log(\Gamma(kx+(1-k)y)) = \log\left(\int_0^\infty t^{kx+(1-k)y-1}e^{-t}dt\right) \qquad x, y \in (0,\infty)$$
$$= \log\left(\int_0^\infty t^{kx-k+k+(1-k)y-1}e^{-kt+kt-t}dt\right)$$
$$= \log\left(\int_0^\infty t^{kx-k}e^{-kt}\,t^{k+(1-k)y-1}e^{kt-t}dt\right)$$
$$= \log\left(\int_0^\infty (t^{x-1}e^{-t})^k (t^{y-1}e^{-t})^{1-k}dt\right)$$

Holder's inequality

$$\int_a^b |f(x)g(x)|dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}} \quad \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

$$\int_0^\infty (t^{x-1}e^{-t})^k (t^{y-1}e^{-t})^{1-k} dt \le \left(\int_0^\infty t^{x-1}e^{-t}dt\right)^k \left(\int_0^\infty t^{y-1}e^{-t}dt\right)^{1-k}$$

$$\log(\Gamma(kx+(1-k)y)) \le \log\left\{\left(\int_0^\infty t^{x-1}e^{-t}dt\right)^k \left(\int_0^\infty t^{y-1}e^{-t}dt\right)^{1-k}\right\}$$
$$= k\log\Gamma(x) + (1-k)\log\Gamma(y)$$

$\log\Gamma(x)$ is a convex function $\Rightarrow$ $\Gamma(x)$ is a log-convex function.

Proof of Bohr-Mollerup Theorem.

For $\forall x>0,\ n\in\mathbb{N}$

$$f(n+x) = (n+x-1)f(n+x-1) = (n+x-1)(n+x-2)f(n+x-2)$$
$$= (n+x-1)(n+x-2)(n+x-3)\cdots(x+2)(x+1)x f(x)$$
$$[\text{using } f(x+1) = xf(x)]$$

Let's restrict the realm of x to $x\in(0,1)$ and of n to $n\ge 2$.

$f(x)$ is a log-convex function. Thus we can say

$$\frac{\log f(n) - \log f(n-1)}{n-(n-1)} \le \frac{\log f(n+x) - \log f(n)}{(n+x)-n} \le \frac{\log f(n+1) - \log f(n)}{(n+1)-n}$$

Since $f(n) = (n-1)f(n-1) = (n-1)(n-2)f(n-2) = (n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1\cdot f(1)$
$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = (n-1)!$
$(\because f(1)=1)$

$$\frac{\log(n-1)! - \log(n-2)!}{1} \le \frac{\log f(n+x) - \log(n-1)!}{x} \le \frac{\log n! - \log(n-1)!}{1}$$

$$x\log(n-1) \le \log\frac{f(n+x)}{(n-1)!} \le x\log n$$

$$\Rightarrow (n-1)^x (n-1)! \le f(n+x) \le n^x (n-1)!$$

Since $f(n+x) = (n+x-1)(n+x-2)(n+x-3)\cdots(x+2)(x+1)\,x\,f(x)$.

$$\frac{(n-1)^x(n-1)!}{(n+x-1)(n+x-2)(n+x-3)\cdots(x+2)(x+1)x} \le f(x) \le \frac{n^x(n-1)!}{(n+x-1)(n+x-2)(n+x-3)\cdots(x+2)(x+1)x}$$

LHS

$$\lim_{n\to\infty}\frac{(n-1)^x(n-1)!}{(n+x-1)\cdots(x+1)x} = \lim_{n\to\infty}\frac{n^x n!}{(n+x)(n+x-1)(n+x-2)\cdots(x+2)(x+1)x}$$

$(n \to n+1)$

RHS

$$\lim_{n\to\infty}\frac{n^x(n-1)!}{(n+x-1)(n+x-2)\cdots(x+1)x} = \lim_{n\to\infty}\left\{\frac{n^x n!}{(n+x)(n+x-1)\cdots(x+1)x}\times\frac{n+x}{n}\right\}$$

$$= \lim_{n\to\infty}\frac{n^x n!}{(n+x)(n+x-1)(n+x-2)\cdots(x+2)(x+1)x}$$

$$\Rightarrow f(x) = \lim_{n\to\infty}\frac{n^x n!}{(n+x)(n+x-1)(n+x-2)\cdots(x+2)(x+1)x}$$

The value of limit is unique → The value of $f(x)$ $(x\in(0,1])$ is unique.

By $f(x+1) = xf(x)$ The value of $f(x)$ for all $x\in(0,\infty)$ is also unique. $\blacksquare$

$\Gamma(x)$ satisfies all properties of the theorem above $\begin{cases}\Gamma(1)=1\\ \Gamma(x+1)=x\Gamma(x)\\ \text{log-convex}\end{cases}$

Thus, $\Gamma(x)$ is a unique function which satisfies $f(x+1)=xf(x)$, $f(1)=1$, log-convex function)

Thus, we can define factorial of real number with $\Gamma(x)$.

4. Severel Functional Equation of Gamma Function.

By Bohr-Mollerup theorem, we can write Gamma Function like

$$\Gamma(x)=\lim_{n\to\infty}\frac{n^x n!}{(n+x)(n+x-1)(n+x-2)\cdots(x+1)x}$$

Let $\Gamma_r(x)=\dfrac{r^x r!}{(r+x)(r+x-1)(r+x-2)\cdots(x+1)x}$ $\quad \Gamma(x)=\lim_{r\to\infty}\Gamma_r(x)$

$$\Gamma_r(x)=\frac{e^{x\ln r}r!}{(r+x)(r+x-1)(r+x-2)\cdots(x+1)x}$$

$$=\frac{e^{x\ln r}}{\left(1+\frac{x}{r}\right)\left(1+\frac{x}{r-1}\right)\left(1+\frac{x}{r-2}\right)\cdots\left(1+\frac{1}{x}\right)x}$$

$$=\frac{e^{x(\ln r-1-1/2-1/3\cdots 1/r)}e^{(x+x/2+x/3+\cdots x/r)}}{\left(1+\frac{x}{r}\right)\left(1+\frac{x}{r-1}\right)\left(1+\frac{x}{r-2}\right)\cdots(1+x)x}$$

$$=\frac{e^{x(\ln r-H_r)}}{x}\cdot\frac{e^x}{1+x}\cdot\frac{e^{x/2}}{1+\frac{x}{2}}\cdot\frac{e^{x/3}}{1+\frac{x}{3}}\cdots\frac{e^{x/r}}{1+\frac{x}{r}}$$

$$=\frac{e^{-x(H_r-\ln r)}}{x}\prod_{k=1}^{r}\frac{e^{x/k}}{1+\frac{x}{k}}$$

$$\frac{1}{\Gamma(x)}=\frac{1}{\lim_{r\to\infty}\Gamma_r(x)}=\lim_{r\to\infty}xe^{x(H_r-\ln r)}\prod_{k=1}^{r}\frac{1+\frac{x}{k}}{e^{x/k}}$$

$$\frac{1}{\Gamma(x)}\cdot\frac{1}{\Gamma(-x)}=\lim_{r\to\infty} x e^{x(H_r-\ln r)}\prod_{k=1}^{r}\frac{1+\frac{x}{k}}{e^{x/k}}(-x)e^{-x(H_r-\ln r)}\prod_{k=1}^{r}\frac{1-\frac{x}{k}}{e^{-x/k}}$$

$$=-x^2\lim_{r\to\infty}\prod_{k=1}^{r}\left(1-\frac{x^2}{k^2}\right)=-x^2\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$$

$$\sin x = x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right) \Rightarrow \sin\pi x=\pi x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$$

$$\Rightarrow -x^2\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)=-\frac{x}{\pi}\sin\pi x$$

$$\Rightarrow \frac{1}{\Gamma(x)\Gamma(-x)}=-\frac{x}{\pi}\sin\pi x$$

$$\Gamma(-x)=\frac{\Gamma(1-x)}{-x} \Rightarrow \boxed{\frac{1}{\Gamma(x)\Gamma(1-x)}=\frac{\sin\pi x}{\pi}} //$$

$$\boxed{B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}$$

<u>proof</u>

$$\Gamma(m)=\int_0^\infty t^{m-1}e^{-t}dt$$

$$t=zx \quad (z>0) \quad dt=z\,dx \qquad \begin{array}{l} t=0 \Rightarrow x=0 \\ t\to\infty \Rightarrow x\to\infty \end{array}$$

$$\Rightarrow \Gamma(m)=\int_0^\infty (zx)^{m-1}e^{-zx}z\,dx=\int_0^\infty z^m x^{m-1}e^{-zx}dx$$

$$\Gamma(n)=\int_0^\infty z^{n-1}e^{-z}dz$$

$$\Gamma(m)\Gamma(n)=\int_0^\infty z^m x^{m-1}e^{-zx}dx\int_0^\infty z^{n-1}e^{-z}dz$$

$$=\int_0^\infty\int_0^\infty z^{m+n-1}x^{m-1}e^{-(x+1)z}dz\,dx$$

$$(x+1)z=y \quad (x+1)dz=dy \quad dz=\frac{dy}{x+1}$$

$$z=0 \Rightarrow y=0,\ z\to\infty \Rightarrow y\to\infty$$

$$\Gamma(m)\Gamma(n)=\int_0^\infty\int_0^\infty \frac{y^{m+n-1}}{(x+1)^{m+n-1}}x^{m-1}e^{-y}\frac{dy}{x+1}dx$$

$$=\int_0^\infty\int_0^\infty y^{m+n-1}e^{-y}\frac{x^{m-1}}{(x+1)^{m+n}}dy\,dx$$

$$=\int_0^\infty x^{m-1}(x+1)^{-m-n}\int_0^\infty y^{m+n-1}e^{-y}dy\,dx$$

$$=\left(\int_0^\infty x^{m-1}(x+1)^{-m-n}dx\right)\Gamma(m+n)$$

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\int_0^\infty x^{m-1}(x+1)^{-m-n}dx$$

$$x+1=\frac{1}{1-r} \quad dx=\frac{1}{(1-r)^2}dr, \quad \begin{matrix} x=0 \Rightarrow r=0 \\ x\to\infty \Rightarrow r\to 1^- \end{matrix}$$

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\int_0^1\left(\frac{1}{1-r}-1\right)^{m-1}\left(\frac{1}{1-r}\right)^{-m-n}\left(\frac{1}{(1-r)^2}dr\right)$$

$$=\int_0^1\frac{r^{m-1}}{(1-r)^{m-1}}(1-r)^{m+n-2}dr=\int_0^1 r^{m-1}(1-r)^{n-1}dr$$

$$\Rightarrow \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) = \int_0^1 r^{m-1}(1-r)^{n-1}\,dr \quad ▨$$

//

$$\boxed{\frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(s+\tfrac{1}{2}\right) = \Gamma(2s).}$$

proof

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

Let $x=s,\ y=s,\ t=\frac{z+1}{2}$. $dt = \frac{1}{2}dz$ $\quad t=0 \rightarrow z=-1$, $\ t=1 \rightarrow z=1$

$$\frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = \int_{-1}^1 \left(\frac{z+1}{2}\right)^{s-1}\left(1-\frac{z+1}{2}\right)^{s-1}\frac{1}{2}\,dz$$

$$= \int_{-1}^1 (z+1)^{s-1}(1-z)^{s-1}\frac{1}{2^{2s-1}}\,dz$$

$$= \frac{1}{2^{2s-1}}\int_{-1}^1 (1-z^2)^{s-1}\,dz$$

$$\int_{-1}^1 (1-z^2)^{s-1}\,dz = \int_{-1}^0 (1-z^2)^{s-1}\,dz + \int_0^1 (1-z^2)^{s-1}\,dz$$

Let $z^2 = w$ $\quad 2z\,dz = dw$ $\quad dz = \frac{dw}{2z}$ $\quad \left\langle \frac{dw}{2\sqrt{w}},\ \frac{dw}{-2\sqrt{w}} \right.$ $\quad z=-1 \rightarrow w=1$, $\ z=0 \rightarrow w=0$, $\ z=1 \rightarrow w=1$

$$\int_{-1}^1 (1-z^2)^{s-1}\,dz = \int_1^0 (1-w)^{s-1}\frac{dw}{-2\sqrt{w}} + \int_0^1 (1-w)^{s-1}\frac{dw}{2\sqrt{w}}$$

$$= \int_0^1 (1-w)^{s-1}\frac{dw}{2\sqrt{w}} + \int_0^1 (1-w)^{s-1}\frac{dw}{2\sqrt{w}}$$

$$= \int_0^1 w^{-\frac{1}{2}}(1-w)^{s-1}\,dw = B\left(\tfrac{1}{2}, s\right)$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(\frac{1}{2}+s)} \implies \frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = \frac{1}{2^{2s-1}}\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(\frac{1}{2}+s)}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \frac{\Gamma(\frac{3}{2})}{\frac{1}{2}} = 2\Gamma\left(\tfrac{3}{2}\right) = 2\int_0^\infty t^{\frac{1}{2}}e^{-t}\,dt \qquad t = r^2,\quad dt = 2r\,dr$$

$$= 2\int_0^\infty r e^{-r^2} 2r\,dr = -2\int_0^\infty r(-2re^{-r^2})\,dr$$

Let $-2re^{-r^2} = dv$, $r = u$. then $e^{-r^2} = v$, $dr = du$

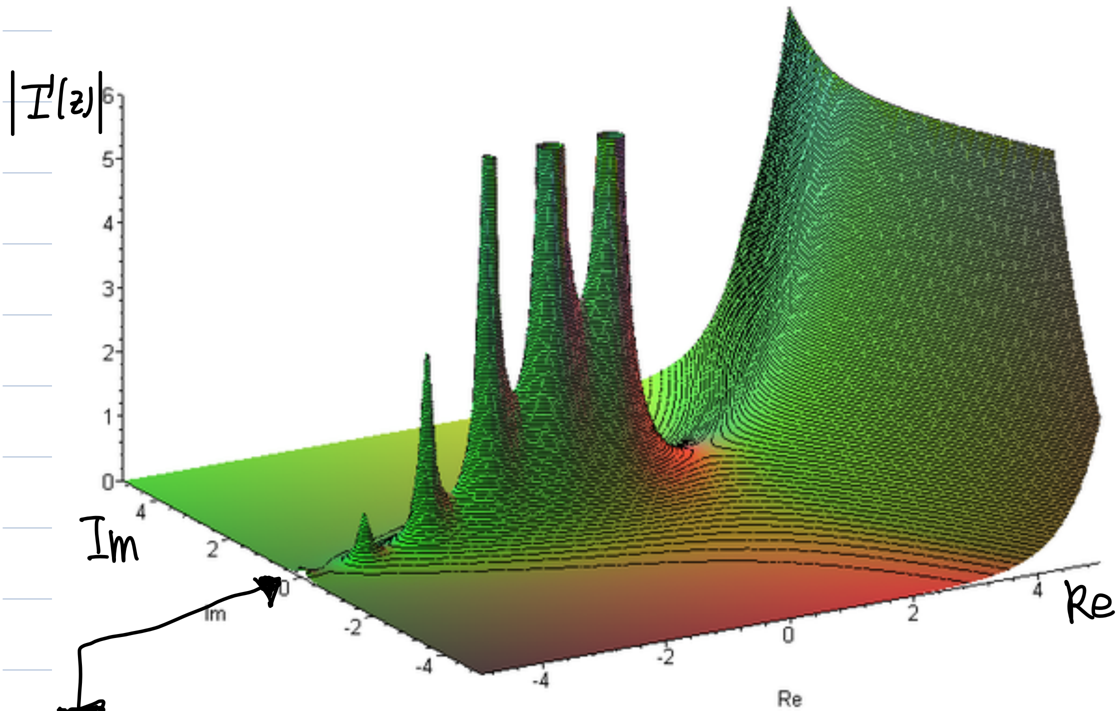
$$= -2\left(\left[re^{-r^2}\right]_0^\infty - \int_0^\infty e^{-r^2}\,dr\right)$$

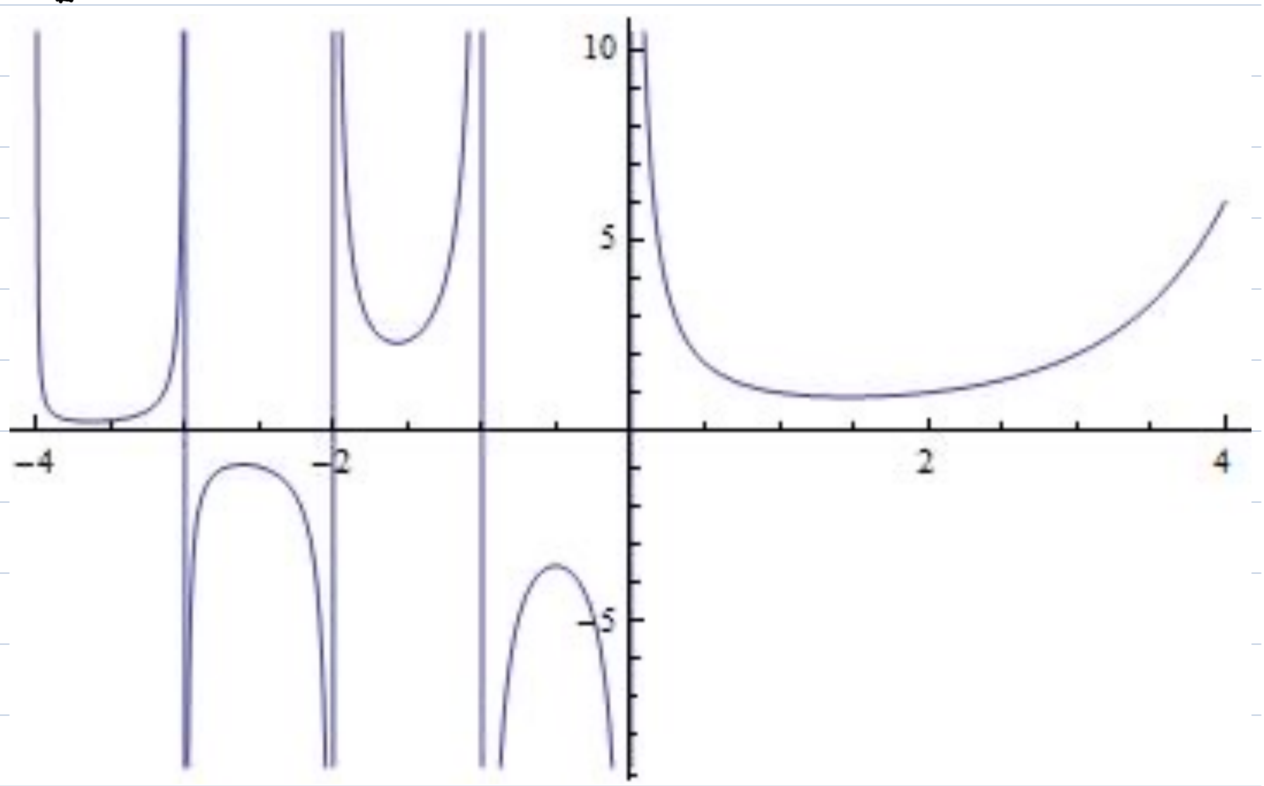
$$= 2\int_0^\infty e^{-r^2}\,dr = \int_{-\infty}^\infty e^{-r^2}\,dr = \sqrt{\pi}$$

$$\frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = \frac{1}{2^{2s-1}}\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(\frac{1}{2}+s)} = \frac{\sqrt{\pi}}{2^{2s-1}}\cdot\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+s)}$$

$$\implies \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(s+\tfrac{1}{2}\right) = \Gamma(2s). \qquad \blacksquare$$

5. The Graph of Gamma Function.





$$\hookrightarrow \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} = \pi \csc \pi x$$

II. Zeta Function.

1. Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

2. The Domain of Riemann Zeta Function.

$$D_\zeta = \{ z \mid z \in \mathbb{C} \setminus \{1\} \}$$

※ $\zeta(z)$ is absolutely convergent when $Re(z) > 1$

Proof

$$|\zeta(z)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^z} \right| \le \sum_{n=1}^{\infty} \left| \frac{1}{n^z} \right|$$

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^z} \right| = \sum_{n=1}^{\infty} \frac{1}{|n^{Re(z) + i Im(z)}|} = \sum_{n=1}^{\infty} \frac{1}{|n^{Re(z)}|} \frac{1}{|n^{i Im(z)}|} \quad \left(n^{i Im(z)} = e^{i Im(z) \ln(n)} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{|n^{Re(z)}|} = \sum_{n=1}^{\infty} \frac{1}{n^{Re(z)}} = \zeta(Re(z))$$

· Integral test. let $s = Re(z)$

① $s > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \to \int_1^{\infty} \frac{1}{x^s} dx = \frac{1}{1-s} x^{1-s} \Big|_1^{\infty} = \lim_{b \to \infty} \frac{1}{1-s} b^{1-s} - \frac{1}{1-s} = -\frac{1}{1-s}$$

=> Convergence

② $s < 1$

$\lim_{b \to \infty} \frac{1}{1-s} b^{1-s} = \infty$ => Divergence

③ $s=1$ $\int_1^\infty \frac{1}{x^s}dx = \int_1^\infty \frac{1}{x}dx = \ln|x|\Big|_1^\infty = \infty - 0 \Rightarrow$ Divergence.

$Re(z)>1 \Rightarrow \zeta(Re(z))$ is convergence $\Rightarrow |\zeta(z)|$ is absolutely convergence

$(|\zeta(z)| \le \zeta(Re(z)))$

$*$ $\zeta(z)$ is divergence if $Re(z) \le 1$

For $z = a+bi \in \mathbb{C}$ $(a,b \in \mathbb{R})$

$$\frac{1}{n^z} = n^{-z} = n^{-a-bi} = n^{-a}n^{-bi} = n^{-a}e^{-bi\ln n}$$

$$= n^{-a}e^{i(-b\ln(n))} = n^{-a}(\cos(-b\ln(n)) + i\sin(-b\ln(n)))$$

$$= n^{-a}(\cos(b\ln(n)) - i\sin(b\ln(n)))$$

$$\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z} = \sum_{n=1}^{\infty} n^{-a}(\cos(b\ln(n)) - i\sin(b\ln(n)))$$

$$Re(\zeta(z)) = \sum_{n=1}^{\infty}\frac{\cos(b\ln(n))}{n^a} = \sum_{n=1}^{\infty}\frac{1}{n^a}\sum_{k=0}^{\infty}\frac{(-1)^k(b\ln(n))^{2k}}{(2k)!} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k b^{2k}}{(2k)!}\frac{(\ln(n))^{2k}}{n^a}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k b^{2k}}{(2k)!}\sum_{n=1}^{\infty}\frac{(\ln(n))^{2k}}{n^a}$$

$$\sum_{n=1}^{\infty}\frac{(\ln(n))^{2k}}{n^a} \xrightarrow[\text{Integral test}]{} \int_1^\infty \frac{(\ln x)^{2k}}{x^a}dx > \int_1^\infty \frac{1}{x^a}dx$$

i) $a=1$ $\int_1^\infty \frac{1}{x}dx = \ln|x|\Big|_1^\infty = \infty - 0 \Rightarrow \sum_{n=1}^{\infty}\frac{(\ln(n))^{2k}}{n^a} \Rightarrow$ Divergence

ii) $a<1$ $\frac{1}{1-a}x^{1-a}\Big|_1^\infty = \infty - \frac{1}{1-a} \Rightarrow$

$Re(\zeta(z))$ is divergence $\Rightarrow \zeta(z)$ is also divergence $\boxed{\text{J.H.W}}$

However, we can define zeta function if $Re(z)<1$ (except $z=1$) by analytic continuation.

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt \rightarrow \Gamma\left(\frac{s}{2}\right)=\int_0^\infty t^{\frac{s}{2}-1}e^{-t}dt$$

$t=\pi n^2 x \quad dt=\pi n^2 dx \quad (n\in\mathbb{N}) \quad \begin{array}{l} t=0 \Rightarrow x=0 \\ t\to\infty \Rightarrow x\to\infty \end{array}$

$$\Gamma\left(\frac{s}{2}\right)=\int_0^\infty (\pi n^2 x)^{\frac{s}{2}-1}e^{-\pi n^2 x}\pi n^2 dx=\int_0^\infty \pi^{\frac{s}{2}} n^s x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$=\pi^{\frac{s}{2}}n^s\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\frac{1}{n^s}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)=\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\sum_{n=1}^\infty \frac{1}{n^s}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)=\sum_{n=1}^\infty\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^\infty e^{-\pi n^2 x}dx$$

$\vartheta(x)=\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$: Jacobi theta function.

$\vartheta(x)=\frac{1}{\sqrt{x}}\vartheta\left(\frac{1}{x}\right)$: Jacobi theta functional equation

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}=\overset{n=0}{1}+2\sum_{n=1}^\infty e^{-\pi n^2 x}=1+2\psi(x)$$

$$\left(\psi(x)=\sum_{n=1}^\infty e^{-\pi n^2 x}\right)$$

$$\vartheta(x)=1+2\psi(x)$$

$$\vartheta(x)=\frac{1}{\sqrt{x}}\vartheta\left(\frac{1}{x}\right)=\frac{1}{\sqrt{x}}\left(1+2\psi\left(\frac{1}{x}\right)\right)=1+2\psi(x)$$

$$\psi(x)=\frac{1}{2\sqrt{x}}+\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)-\frac{1}{2}$$

$$\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}dx=\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$=\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx+\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx=\int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{2\sqrt{x}}+\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)-\frac{1}{2}\right)dx$$

$$=\int_0^1 \frac{1}{2}x^{\frac{s}{2}-\frac{3}{2}}+x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)-\frac{1}{2}x^{\frac{s}{2}-1}dx$$

$$=\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)+\frac{1}{2}\left(x^{\frac{s}{2}-\frac{3}{2}}-x^{\frac{s}{2}-1}\right)dx$$

$$=\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{2}\left(\frac{1}{\frac{s}{2}-\frac{1}{2}}x^{\frac{s}{2}-\frac{1}{2}}-\frac{1}{\frac{s}{2}}x^{\frac{s}{2}}\right)\Bigg|_0^1$$

$$=\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{2}\left(\frac{2}{s-1}-\frac{2}{s}\right)$$

$$=\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{s-1}-\frac{1}{s}$$

Let $x=\frac{1}{y}$ $\quad dx=\frac{-1}{y^2}dy$ $\quad x\to 0^+ \Rightarrow y\to\infty$

$x=1 \Rightarrow y=1$

$$=\int_\infty^1 y^{-\frac{s}{2}+\frac{3}{2}}\psi(y)\left(-\frac{1}{y^2}dy\right)+\frac{1}{s-1}-\frac{1}{s}$$

$$=\int_1^\infty y^{-\frac{s}{2}-\frac{1}{2}}\psi(y)\,dy+\frac{1}{s-1}-\frac{1}{s}$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}dx$$

$$=\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx+\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$=\int_1^\infty y^{-\frac{s}{2}-\frac{1}{2}}\psi(y)dy+\frac{1}{s-1}-\frac{1}{s}+\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$=\int_1^\infty \left(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(x)\,dx+\frac{1}{s-1}-\frac{1}{s}$$

$s \to 1-s$

$$\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)=\int_1^\infty\left(x^{-\frac{s}{2}-\frac{1}{2}}+x^{\frac{s}{2}-1}\right)\psi(x)\,dx+\frac{1}{-s}-\frac{1}{1-s}$$

$$=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\therefore\ \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \Rightarrow \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)=\Gamma\left(\frac{s}{2}\right)\zeta(s). \quad \blacksquare$$

- $\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)=\Gamma\left(\frac{s}{2}\right)\zeta(s) \rightarrow$ eq.1
- $\Gamma(s)\Gamma(1-s)=\pi/\sin\pi s \rightarrow$ eq.2
- $\dfrac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(\frac{1}{2}+s\right)=\Gamma(2s) \rightarrow$ eq.3

eq.1

$$\zeta(s)=\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\frac{1}{\Gamma\left(\frac{s}{2}\right)}$$

eq.3 $s \to \frac{1-s}{2}$

$$\frac{2^{-s}}{\sqrt{\pi}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)=\Gamma(1-s)$$

$$\Gamma\left(\frac{1-s}{2}\right)=2^s\pi^{\frac{1}{2}}\Gamma(1-s)\frac{1}{\Gamma\left(1-\frac{s}{2}\right)}$$

$$\zeta(s)=\pi^{s-\frac{1}{2}}\left(2^s\pi^{\frac{1}{2}}\Gamma(1-s)\frac{1}{\Gamma\left(1-\frac{s}{2}\right)}\right)\zeta(1-s)\frac{1}{\Gamma\left(\frac{s}{2}\right)}$$

$$=2^s\pi^s\Gamma(1-s)\frac{1}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)}\zeta(1-s)$$

eq.2 $s \to \frac{s}{2}$

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)=\pi/\sin\frac{\pi}{2}s$$

$$\zeta(s)=2^s\pi^s\Gamma(1-s)\frac{\sin\frac{\pi}{2}s}{\pi}\zeta(1-s)=2^s\pi^{s-1}\sin\frac{\pi}{2}s\,\Gamma(1-s)\zeta(1-s).$$

$$\therefore\ \zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi}{2}s\ \Gamma(1-s)\,\zeta(1-s).$$

↳ Analytic continuation

Function	2^s	π^{s-1}	$\sin\frac{\pi}{2}s$	$\Gamma(1-s)$	$\zeta(1-s)$
Domain	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\setminus\{1,2,3,\cdots\}$	$Re(s)<0$

$$\sin\frac{\pi}{2}s \longrightarrow \frac{e^{i\frac{\pi}{2}s} - e^{-i\frac{\pi}{2}s}}{2i}$$

We can use the functional equation when $Re(s)<0$

$\Longrightarrow$ We can define zeta function at $Re(s)<0$

$\Longrightarrow$ $\boxed{\text{The domain of } \zeta(1-s) \text{ changes to } \{s \mid Re(s)<0 \text{ or } Re(s)>1\}}$

$\Longrightarrow$ We can use the function equation when $Re(s)<0$ or $Re(s)>1$.

But! if $s = 2, 3, 4, \cdots$ $\Gamma(1-s)$ is not defined.

$$\zeta(m) = 2^m \pi^{m-1} \sin\frac{\pi}{2}m\ \Gamma(1-m)\,\zeta(1-m) \quad (m = 2,3,4,\cdots)$$

i) m: even

$\sin\frac{\pi}{2}m = 0$, $\Gamma(1-s)$: not defined

$$\lim_{s\to m} \sin\frac{\pi}{2}s\ \Gamma(1-s) = \lim_{s\to m} \sin\frac{\pi}{2}s\ \frac{\pi}{\sin\pi s}\,\frac{1}{\Gamma(s)} \qquad \left(\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}\right)$$

$$= \frac{\pi}{\Gamma(m)} \lim_{s\to m} \frac{\sin\frac{\pi}{2}s}{\sin\pi s} = \frac{\pi}{\Gamma(m)} \lim_{s\to m} \frac{\overset{\to -1 \text{ or } 1}{\frac{\pi}{2}\cos\frac{\pi}{2}s}}{\underset{\to \pi}{\pi\cos\pi s}} \to \text{Convergence.}$$

(L'Hopital's rule)

ii) m: odd (≥ 3)

$$\zeta(m) = \underset{\neq 0}{2^m}\ \underset{\neq 0}{\pi^{m-1}}\ \underset{\neq 0}{\sin\frac{\pi}{2}m}\ \underset{\text{not defined}}{\Gamma(1-m)}\ \underset{= 0}{\zeta(1-m)} \quad (m:\text{odd})$$

$(\zeta(-2), \zeta(-4), \cdots = 0)$

$$\zeta(-2n) = \underset{\neq 0}{2^{-2n}}\ \underset{\neq 0}{\pi^{-2n-1}}\ \underset{= 0}{\sin(-\pi n)}\ \underset{\neq 0}{\Gamma(1+2n)}\ \underset{\neq 0}{\zeta(1+2n)} = 0$$

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}\,dt \qquad t=ru \ (r\in\mathbb{N}) \qquad dt=r\,du \qquad \begin{array}{l} t=0 \Rightarrow u=0 \\ t\to\infty \Rightarrow u\to\infty \end{array}$$

$$=\int_0^\infty (ru)^{s-1}e^{-ru}\,r\,du=\int_0^\infty r^s u^{s-1}e^{-ru}\,du=r^s\int_0^\infty u^{s-1}e^{-ru}\,du$$

$$\frac{1}{r^s}=\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-ru}\,du \qquad \sum_{r=1}^{\infty}\frac{1}{r^s}=\sum_{r=1}^{\infty}\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-ru}\,du$$

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\sum_{r=1}^{\infty}e^{-ru}\,du=\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\,\frac{e^{-u}}{1-e^{-u}}\,du$$

$$=\frac{1}{\Gamma(s)}\int_0^\infty \frac{u^{s-1}}{e^u-1}\,du \qquad \Rightarrow \Gamma(s)\zeta(s)=\int_0^\infty \frac{u^{s-1}}{e^u-1}\,du$$

$$\lim_{s\to m}\left|\Gamma(1-s)\zeta(1-s)\right|=\lim_{s\to m}\left|\int_0^\infty \frac{u^{-s}}{e^u-1}\,du\right|\leq\int_0^\infty \frac{u}{e^u-1}\,du$$

$$=\int_0^\infty \frac{u\,e^{-u}}{1-e^{-u}}\,du=\int_0^\infty u\sum_{r=1}^{\infty}e^{-ru}\,du$$

$$=\sum_{r=1}^{\infty}\int_0^\infty u e^{-ru}\,du$$

$$ru=t \qquad r\,du=dt \qquad u=0 \Rightarrow t=0\,,\ u\to\infty \Rightarrow t\to\infty$$

$$=\sum_{r=1}^{\infty}\int_0^\infty \frac{t}{r}e^{-t}\,\frac{dt}{r}=\sum_{r=1}^{\infty}\int_0^\infty \frac{1}{r^2}te^{-t}\,dt$$

$$=\sum_{r=1}^{\infty}\frac{1}{r^2}\int_0^\infty te^{-t}\,dt=\zeta(2)\Gamma(2)=\frac{\pi^2}{6}\cdot\frac{\sqrt{\pi}}{2}=\frac{\pi^{\frac{5}{2}}}{12}$$

$$\Rightarrow -\frac{\pi^{\frac{5}{2}}}{12}<\lim_{s\to m}\Gamma(1-s)\zeta(1-s)<\frac{\pi^{\frac{5}{2}}}{12}$$

$\therefore$ $\Gamma(1-s\,\zeta(1-s)$ converges as s goes to m.

$\boxed{\text{J.H.W}}$

Thus, we can use the functional equation when $Re(s)<0$ or $Re(s)>1$

Now, we extended the realm of the domain to $\{z \mid z \in \mathbb{C} \setminus [0,1]\}$.
What if $0 \leq Re(z) \leq 1$?

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{(2n)^z} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^z} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^z} - \sum_{n=1}^{\infty} \frac{1}{(2n)^z} + 2\sum_{n=1}^{\infty} \frac{1}{(2n)^z}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} + 2^{1-z} \sum_{n=1}^{\infty} \frac{1}{n^z} = \eta(z) + 2^{1-s}\zeta(z)$$

$$(1-2^{1-z})\zeta(z) = \eta(z) \quad \zeta(z) = \frac{\eta(z)}{1-2^{1-z}} \quad \left(\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}\right)$$

↳ Dirichlet eta function.

The domain of eta function is $\{z \mid z \in \mathbb{C}, Re(z) > 0\}$

$\therefore$ we can define $\zeta(s)$ at $0 < Re(s) < 1$ (except $s = 1$)

$\because \zeta(1) = \frac{1}{1-2^{1-1}} \eta(1) = \frac{1}{0}\eta(1)$

* What if $Re(s) = 0$?

i) $Im(z) \neq 0$

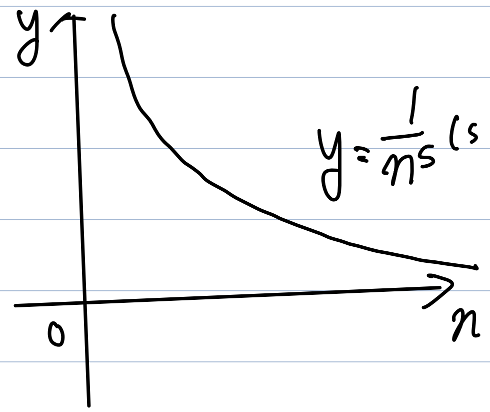
Zeta function can be defined at $0 < Re(s) < 1$
by Dirichlet eta function. Thus we can use the functional equation
below at $s = it$ $(t \in \mathbb{R} \setminus \{0\})$

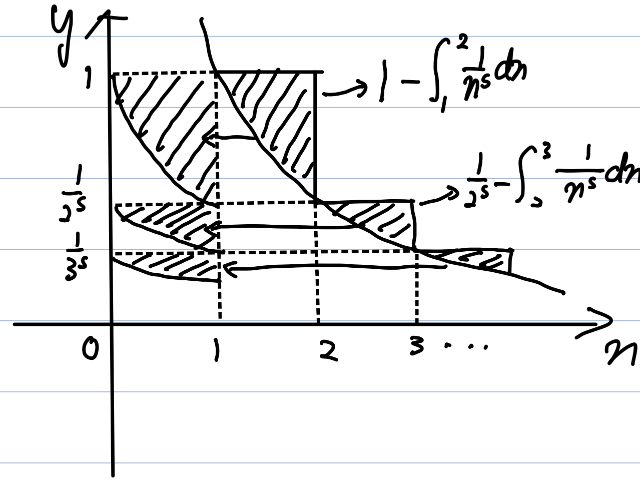
$$\zeta(it) = 2^{it}\pi^{it-1}\sin\left(\frac{\pi}{2}it\right)\Gamma(1-it)\zeta(1-it)$$

what if $s = 0$? $\quad \zeta(0) = 2^0\pi^{0-1}\underbrace{\sin\frac{\pi}{2}0}_{=0}\,\Gamma(1)\underbrace{\zeta(1)}_{\to\infty}$

ii) $s=\sigma$ $(\mathrm{Im}(s)=0)$

\# $\lim_{s\to 1^+}(s-1)\zeta(s)=1$





$$\left(1-\int_1^2 \frac{1}{n^s}dn\right)+\left(\frac{1}{2^s}-\int_2^3 \frac{1}{n^s}dn\right)+\left(\frac{1}{3^s}-\int_3^4 \frac{1}{n^s}dn\right)$$

$$+\cdots+\left(\frac{1}{m^s}-\int_m^{m+1}\frac{1}{n^s}dn\right)=\sum_{n=1}^{m}\frac{1}{n^s}-\int_1^{m+1}\frac{1}{n^s}dn<1\times 1=1$$

$$\lim_{m\to\infty}\left(\sum_{n=1}^{m}\frac{1}{n^s}-\frac{1}{1-s}n^{1-s}\Big|_1^{m+1}\right)=\zeta(s)-\lim_{m\to\infty}\frac{(m+1)^{1-s}}{1-s}+\frac{1}{s-1}$$

$$=\zeta(s)-\frac{1}{s-1}<1$$

$$\lim_{m\to\infty}\left(\sum_{n=1}^{m}\frac{1}{n^s}-\frac{1}{1-s}n^{1-s}\Big|_1^{m+1}\right)\geq 0>-1$$

$$\Rightarrow -1<\zeta(s)-\frac{1}{s-1}<1 \Rightarrow 1-s<(s-1)\zeta(s)-1<s-1$$

$$\Rightarrow 2-s<(s-1)\zeta(s)<s$$

$$\lim_{s\to 1^+}(2-s)\leq \lim_{s\to 1^+}(s-1)\zeta(s)\leq \lim_{s\to 1^+}s \Rightarrow \lim_{s\to 1^+}(s-1)\zeta(s)=1 \quad \blacksquare$$

$$(s-1)\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi}{2}s\,\Gamma(1-s)\zeta(1-s)(s-1) \quad (s>1)$$

$$1 = \lim_{s\to 1^+}(s-1)\zeta(s) = 2^1\pi^0\sin\frac{\pi}{2}\zeta(0)\lim_{s\to 1^+}(s-1)\Gamma(1-s)$$

$$= 2\zeta(0)\lim_{s\to 1^+}\{-\Gamma(2-s)\} = -2\zeta(0)\Gamma(1)$$

$$\Gamma(1) = (1-1)! = 0! = 1$$

$$\therefore 1 = -2\zeta(0) \Rightarrow \zeta(0) = -\frac{1}{2}$$

$$\zeta(1) = 2^1\pi^{1-1}\sin\frac{\pi}{2}\Gamma(1-1)\zeta(1-1) = 2\zeta(0)\Gamma(0) = -\Gamma(0)$$

↳ undefined.

$\zeta(s)$ can't defined at $s=1$ by any method.

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty}\frac{1}{n^s} & (\mathrm{Re}(s)>1) \\ \zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi}{2}s\,\Gamma(1-s)\zeta(1-s) & (s\in\mathbb{C}\setminus\{1\}) \\ \zeta(s) = \frac{1}{1-2^{1-s}}\eta(s) & (\mathrm{Re}(s)>0) \end{cases}$$

$$\therefore \text{Domain of } \zeta = \{z \mid z\in\mathbb{C}\setminus\{1\}\}$$

3. The Value of ζ of Even Number and Negative Integer.

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n-1} B_{2n}}{(2n)!} \quad (n \in \mathbb{N}) \qquad \zeta(-n) = (-1)^{n} \frac{B_{n+1}}{n+1}$$

$$\sin x = x \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right) \quad x \to \pi x \quad \sin \pi x = \pi x \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$$

$$\ln \sin \pi x = \ln\left\{\pi x \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)\right\} = \ln \pi x + \ln \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$$

$$= \ln \pi x + \sum_{k=1}^{\infty} \ln\left(1-\frac{x^2}{k^2}\right)$$

$$\frac{d}{dx}\ln(\sin \pi x) = \frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{-\frac{2x}{k^2}}{1-\frac{x^2}{k^2}} = \frac{1}{x} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{x^2}{k^2}\right)^{n-1}\left(\frac{-2x}{k^2}\right)$$

$$= \frac{1}{x} - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{2n-1}}{k^{2n}} = \frac{1}{x} - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{2n-1}}{k^{2n}}$$

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - 2\sum_{n=1}^{\infty} x^{2n-1}\zeta(2n)$$

$$\frac{\pi x \cos \pi x}{\sin \pi x} = 1 - 2\sum_{n=1}^{\infty}\zeta(2n)\, x^{2n}.$$

$e^{ix} = \cos x + i \sin x$ (Euler's Formula)

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i \sin x$$

$$e^{ix} + e^{-ix} = \cos x + i\sin x + \cos x - i \sin x = 2\cos x \quad \to \cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$e^{ix} - e^{-ix} = \cos x + i\sin x - (\cos x - i\sin x) = 2i\sin x \quad \to \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\frac{\cos x}{\sin x} = \frac{\frac{e^{ix}+e^{-ix}}{2}}{\frac{e^{ix}-e^{-ix}}{2i}} = i\,\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = i\,\frac{e^{2ix}+1}{e^{2ix}-1}$$

$$x \to \pi x \quad \frac{\cos \pi x}{\sin \pi x} = i\,\frac{e^{2i\pi x}+1}{e^{2i\pi x}-1} \quad \to \quad \frac{\pi x \cos \pi x}{\sin \pi x} = i\pi x\,\frac{e^{2i\pi x}+1}{e^{2i\pi x}-1}$$

$$\frac{\pi x\cos\pi x}{\sin\pi x} = i\pi x\left(\frac{e^{2i\pi x}-1+2}{e^{2i\pi x}-1}\right) = i\pi x\left(1+\frac{2}{e^{2i\pi x}-1}\right)$$

$$= i\pi x + \frac{2i\pi x}{e^{2i\pi x}-1}$$

$$\frac{x}{e^{x}-1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}x^m \quad \cdots\cdots (5) \quad (B_m\text{: Bernoulli number})$$

$x \to 2i\pi x$

$$\frac{2i\pi x}{e^{2i\pi x}-1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}(2i\pi x)^m$$

$$= \frac{B_0}{0!}(2i\pi x)^0 + \frac{B_1}{1!}(2i\pi x)^1 + \sum_{m=2}^{\infty}\frac{B_m}{m!}(2i\pi x)^m$$

$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_{2r+1} = 0 \quad (r \in \mathbb{N})$

$$= 1 - i\pi x + \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}(2i\pi x)^{2m}$$

$$\frac{\pi x\cos\pi x}{\sin\pi x} = i\pi x + \frac{2i\pi x}{e^{2i\pi x}-1} = i\pi x + 1 - i\pi x + \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}(2i\pi x)^{2m}$$

$$= 1 + \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}2^{2m}i^{2m}\pi^{2m}x^{2m}$$

$$= 1 - 2\sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}2^{2m-1}(-1)^{m+1}\pi^{2m}x^{2m}$$

$$= 1 - 2\sum_{n=1}^{\infty}\zeta(2n)x^{2n}$$

$$\therefore\ \zeta(2n) = \frac{B_{2n}}{(2n)!}2^{2n-1}\pi^{2n}(-1)^{n+1} = (-1)^{n+1}\frac{2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}$$

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi}{2}s\, \Gamma(1-s)\zeta(1-s)$$

$s = -2n+1 \quad (n \in \mathbb{N})$

$\mapsto \Gamma(x) = (x-1)!$

$$\zeta(-2n+1) = 2^{-2n+1}\pi^{-2n}\sin\left(\frac{\pi}{2}(-2n+1)\right)\Gamma(2n)\zeta(2n)$$

$$= 2^{-2n+1}\pi^{-2n}(-1)^n(2n-1)!\,(-1)^{n+1}\frac{2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}$$

$$= 2^{-2n+1+2n-1}\pi^{-2n+2n}(-1)^{2n+1}\frac{(2n-1)!}{(2n)!}B_{2n}$$

$$= -\frac{1}{2n}B_{2n}$$

$$\zeta(-2n) = 2^{-2n}\pi^{-2n-1}\underset{\overset{\shortparallel}{0}}{\sin(-\pi n)}\ \underset{\ne 0}{\Gamma(1+2n)}\ \underset{\ne 0}{\zeta(1+2n)} = 0$$

$$\zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1}$$

4. The Graph of Zeta Function.

① Cartesian coordinates system

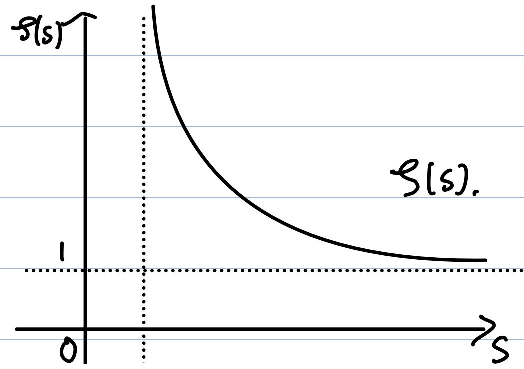
i) $s > 1$

- $\lim_{s\to 1+}\zeta(s) = \lim_{s\to 1+}\sum_{n=1}^{\infty}\frac{1}{n^s} = \infty$ • $\lim_{s\to\infty}\zeta(s) = \lim_{s\to\infty}\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right) = 1$

- $\frac{d}{ds}\zeta(s) = \frac{d}{ds}\sum_{n=1}^{\infty}\frac{1}{n^s} = -s\sum_{n=1}^{\infty}\frac{1}{n^{s+1}} < 0 \implies$ decreasing

- $\frac{d^2}{ds^2}\zeta(s) = \frac{d}{ds}\left(-s\sum_{n=1}^{\infty}\frac{1}{n^{s+1}}\right) = s(s+1)\sum_{n=1}^{\infty}\frac{1}{n^{s+2}} > 0 \implies$ concaved downward

- $\zeta(s)$ is continuous at $s > 1$

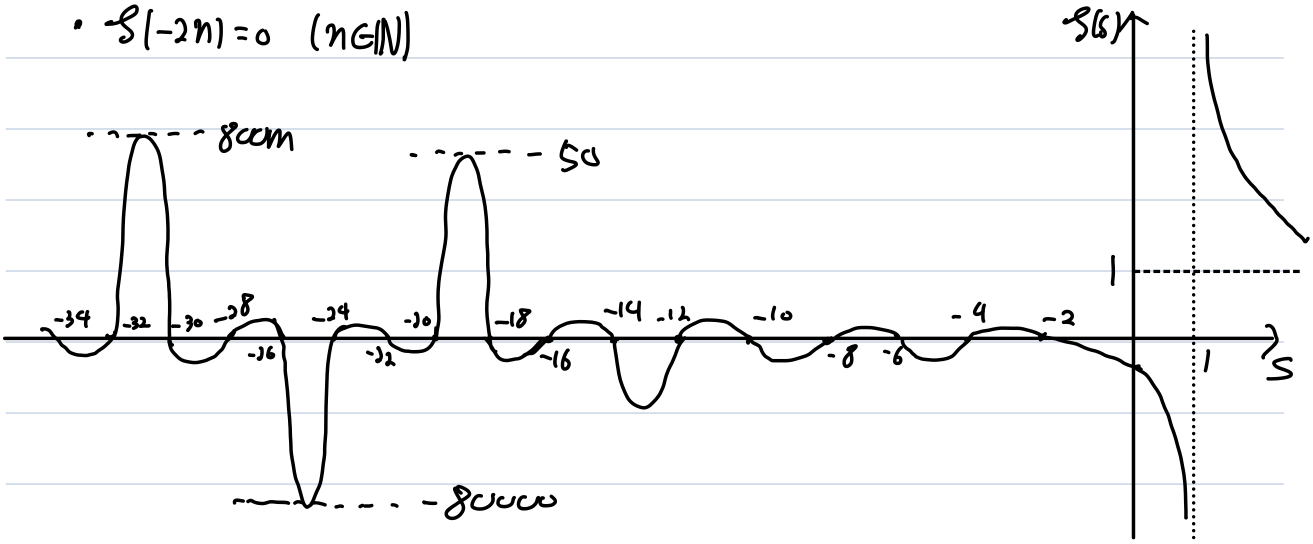


ii) $s<1$

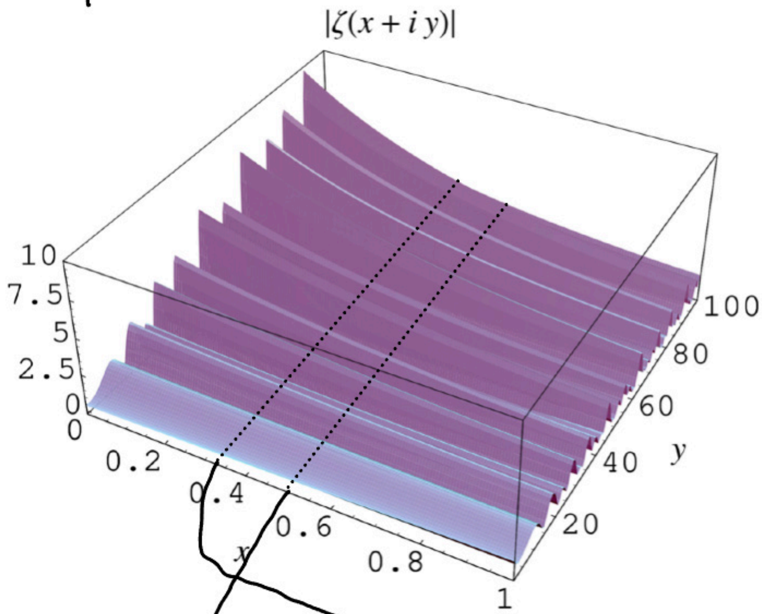
- $\lim_{s\to1^-}\zeta(s)=\lim_{s\to1^-}\frac{\eta(s)}{1-2^{1-s}}$ $\quad\lim_{s\to1^-}\frac{\ln 2}{1-2^{1-s}}=-\infty$ $\quad\lim_{s\to1^-}\eta(s)=\eta(1)=\ln 2$

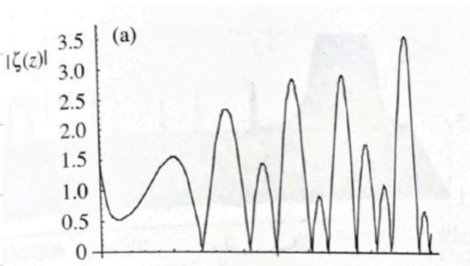
  $=-\infty$

- $\zeta(0)=-\frac{1}{2}$
- $\zeta(-2n)=0 \quad (n\in\mathbb{N})$

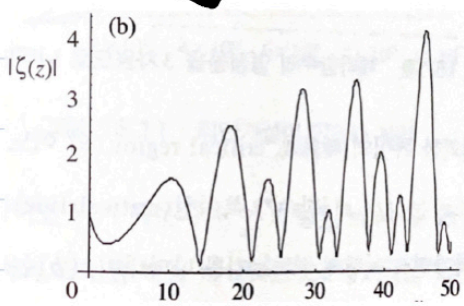


② Complex plane





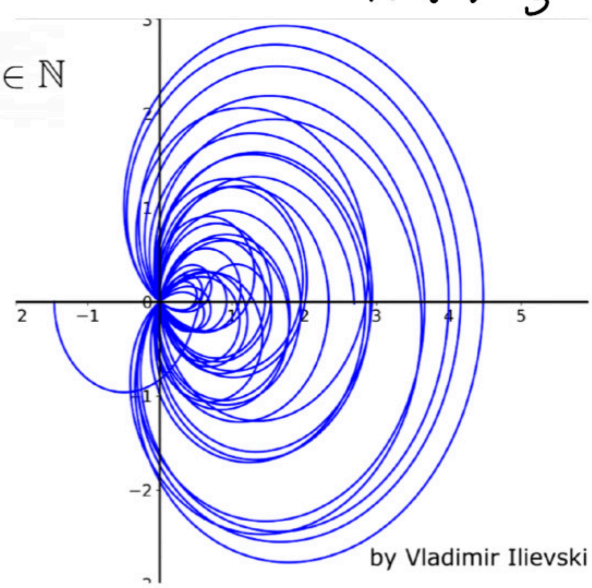
$Re(z) = \frac{1}{2}$



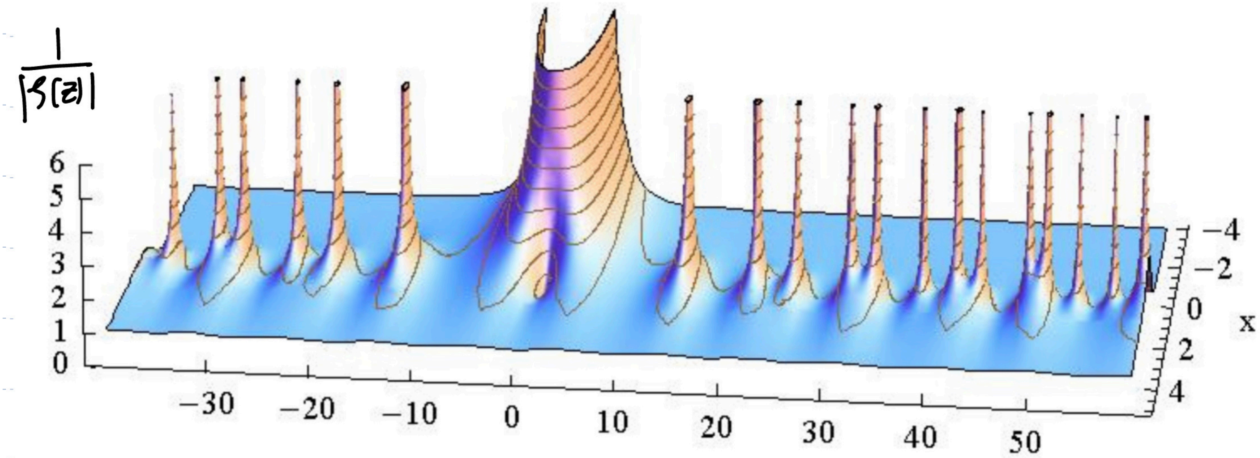
$Re(z) = \frac{1}{3}$

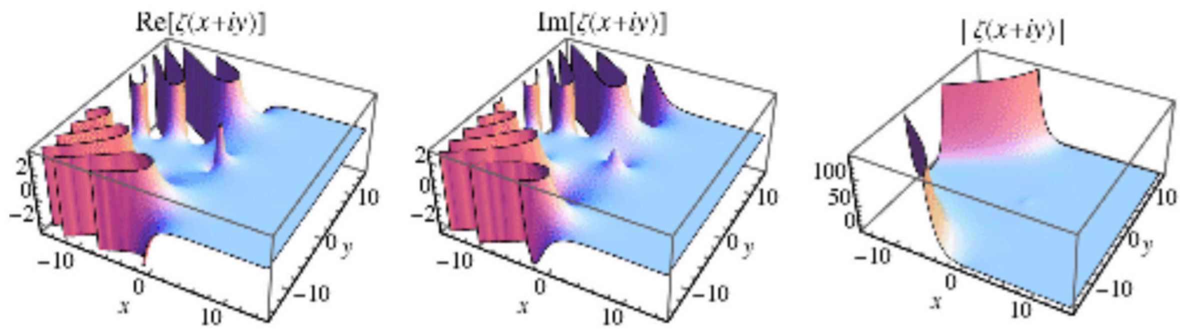
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s \in \mathbb{C}, n \in \mathbb{N}$$

$$\zeta(s) = 0, \forall s = \frac{1}{2} + ti$$



python





5. The zero points of Zeta Function.

$\zeta(-2n)=0 \implies -2n$: trivial zero point

if $s>1$,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\frac{1}{9^s}+\frac{1}{10^s}+\frac{1}{10^s}+\frac{1}{11^s}+\frac{1}{12^s}+\cdots$$

$$<1+\frac{1}{2^s}+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{4^s}+\frac{1}{4^s}+\frac{1}{4^s}+\frac{1}{8^s}+\frac{1}{8^s}+\frac{1}{8^s}+\frac{1}{8^s}+\frac{1}{8^s}+\frac{1}{8^s}+\frac{1}{8^s}+\frac{1}{8^s}\cdots$$

$$=1+\frac{2}{2^s}+\frac{4}{4^s}+\frac{8}{8^s}+\frac{16}{16^s}+\cdots$$

$$=1+\frac{1}{2^{s-1}}+\frac{1}{4^{s-1}}+\frac{1}{8^{s-1}}+\frac{1}{16^{s-1}}+\cdots$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2^{s-1}}\right)^{n-1}$$

$$=\frac{1}{1-\frac{1}{2^{s-1}}}=\frac{1}{1-2^{1-s}}$$

$$|\zeta(z)| < \zeta(\mathrm{Re}(z)) < \frac{1}{1-2^{1-\mathrm{Re}(z)}} \qquad (\mathrm{Re}(z)>1)$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{r_1, r_2, r_3, \cdots \geq 0} \frac{1}{2^{r_1 s} 3^{r_2 s} 5^{r_3 s} \cdots}$$

$$= \sum_{r_1=0}^{\infty} \left(\frac{1}{2^s}\right)^{r_1} \sum_{r_2=0}^{\infty} \left(\frac{1}{3^s}\right)^{r_2} \sum_{r_3=0}^{\infty} \left(\frac{1}{5^s}\right)^{r_3} \cdots$$

$$= \frac{1}{1-\left(\frac{1}{2}\right)^s} \cdot \frac{1}{1-\left(\frac{1}{3}\right)^s} \cdot \frac{1}{1-\left(\frac{1}{5}\right)^s} \cdots = \prod_{p \in \mathbb{P}}^{\infty} \frac{1}{1-p^{-s}}$$

$$\zeta(z) = \prod_{p \in \mathbb{P}}^{\infty} (1-p^{-z})^{-1}$$

$$\frac{1}{|\zeta(z)|} = \left| \frac{1}{\prod_{p \in \mathbb{P}}^{\infty} (1-p^{-z})^{-1}} \right| = \prod_{p \in \mathbb{P}}^{\infty} |1-p^{-z}| = \prod_{p \in \mathbb{P}}^{\infty} \left| \frac{p^z-1}{p^z} \right| < \prod_{p \in \mathbb{P}}^{\infty} \left| \frac{p^z}{p^z-1} \right|$$

$$= \prod_{p \in \mathbb{P}}^{\infty} \left| \frac{1}{1-p^{-z}} \right| = \left| \prod_{p \in \mathbb{P}}^{\infty} (1-p^{-z})^{-1} \right| = |\zeta(z)| < \frac{1}{1-2^{1-\mathrm{Re}(z)}}$$

$$\Rightarrow |\zeta(z)| > 1-2^{1-\mathrm{Re}(z)} > 0$$

$\therefore$ There are no zero points when $\mathrm{Re}(z)$ is larger than 1.

What if $\mathrm{Re}(z)=1$? $(\mathrm{Im}(z) \neq 0)$

$$\zeta(z) = \prod_{p \in \mathbb{P}}^{\infty} (1-p^{-z})^{-1} \qquad (\mathrm{Re}(z)>1)$$

$$|\zeta(z)| = \left| \prod_{p \in \mathbb{P}}^{\infty} (1-p^{-z})^{-1} \right| = \prod_{p \in \mathbb{P}}^{\infty} |1-p^{-z}|^{-1}$$

$$\ln|\zeta(z)| = \ln\left(\prod_{p\in\mathbb{P}}^{\infty}|1-p^{-z}|^{-1}\right) = -\sum_{p\in\mathbb{P}}^{\infty}\ln|1-p^{-z}|$$

$w = re^{i\theta}$ $(r=|w|,\ \theta=\arg(w))$

$\ln w = \ln re^{i\theta} = \ln r + i\theta \implies \mathrm{Re}(\ln w) = \ln r = \ln|w| \quad \therefore \ln|w| = \mathrm{Re}(\ln w)$

$$\ln|\zeta(z)| = -\sum_{p\in\mathbb{P}}^{\infty}\ln|1-p^{-z}| = -\mathrm{Re}\sum_{p\in\mathbb{P}}^{\infty}\ln(1-p^{-z}) = \mathrm{Re}\left(\sum_{p\in\mathbb{P}}^{\infty}-\ln(1-p^{-z})\right)$$

$$1+x+x^2+\cdots = \frac{1}{1-x} \implies -\ln|1-x| = x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots = \sum_{n=1}^{\infty}\frac{x^n}{n}$$

$(C=0)$

$$\ln|\zeta(z)| = \mathrm{Re}\left(\sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty}\frac{p^{-nz}}{n}\right)$$

Let $z = a+bi$ $(a,b\in\mathbb{R},\ a=\mathrm{Re}(z)>1)$

$p^{-nz} = p^{-na-nbi} = p^{-na}\times p^{-nbi}$

$p^{-nbi} = e^{-nbi\ln p} = e^{i(-bn\ln p)} = \cos(-bn\ln p) + i\sin(-bn\ln p)$

$= \cos(bn\ln p) - i\sin(bn\ln p)$

$\mathrm{Re}(p^{-nz}) = p^{-na}\cos(bn\ln p) \implies \mathrm{Re}\left(\frac{p^{-nz}}{n}\right) = \frac{p^{-na}}{n}\cos(bn\ln p)$

$$\ln|\zeta(z)| = \sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty}\mathrm{Re}\left(\frac{p^{-nz}}{n}\right) = \sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty}\frac{p^{-na}\cos(bn\ln p)}{n} = \sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty}\frac{\cos(b\ln p^n)}{np^{na}}$$

$3\ln|\zeta(a+0i)| + 4\log|\zeta(a+bi)| + \log(\zeta(a+2bi))$

$$= \sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty}\frac{3\cos(0\cdot\ln p^n)}{np^{na}} + \sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty}\frac{4\cos(b\ln p^n)}{np^{na}} + \sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty}\frac{\cos(2b\ln p^n)}{np^{na}}$$

$$= \sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty}\frac{3+4\cos(b\ln p^n)+\cos(2b\ln p^n)}{np^{na}}$$

$$3+4\cos\theta+\cos 2\theta = 2\cos^2\theta - 1 + 4\cos\theta + 3$$

$$= 2\cos^2\theta + 4\cos\theta + 2 = 2(\cos\theta + 1)^2 \geq 0$$

$$\theta = b\ln p^n \Rightarrow 3 + 4\cos(b\ln p^n) + \cos(2b\ln p^n) \geq 0$$

$$\Rightarrow \sum_{p\in\mathbb{P}}^{\infty}\sum_{n=1}^{\infty} \frac{3+4\cos(b\ln p^n)+\cos(2b\ln p^n)}{n p^{na}} \geq 0$$

$$\Rightarrow 3\ln|\zeta(a)| + 4\ln|\zeta(a+bi)| + \ln|\zeta(a+2bi)| \geq 0$$

$$\Rightarrow \ln|\zeta(a)|^3|\zeta(a+bi)|^4|\zeta(a+2bi)| \geq 0$$

$$\Rightarrow |\zeta(a)|^3|\zeta(a+bi)|^4|\zeta(a+2bi)| \geq 1$$

$$\lim_{a\to 1^+}\left\{|\zeta(a)|^3|\zeta(a+bi)|^4|\zeta(a+2bi)|\right\} \geq 1$$

$$= \lim_{a\to 1^+}\left\{|(a-1)\zeta(a)|^3\left|\frac{\zeta(a+bi)}{a-1}\right|^4 |a-1||\zeta(a+2bi)|\right\}$$

Assume that $\zeta(1+bi)=0$ at $b=b_0$.

$$= \lim_{a\to 1^+}\left\{|(a-1)\zeta(a)|^3\left|\frac{\zeta(a+bi)-\zeta(1+bi)}{a-1}\right|^4 |a-1||\zeta(a+2bi)|\right\}$$

$$= 1^3\cdot \frac{\partial}{\partial a}\zeta(a+bi)\Big|_{a=1} |\zeta(1+2bi)| \lim_{a\to 1^+}|a-1| \geq 1$$

$$\Rightarrow \quad 0 \geq 1$$

$\Rightarrow$ There is no b_0 such that $\zeta(1+b_0 i)=0$

$\Rightarrow$ There is no zero point of ζ at $Re(z)=1$

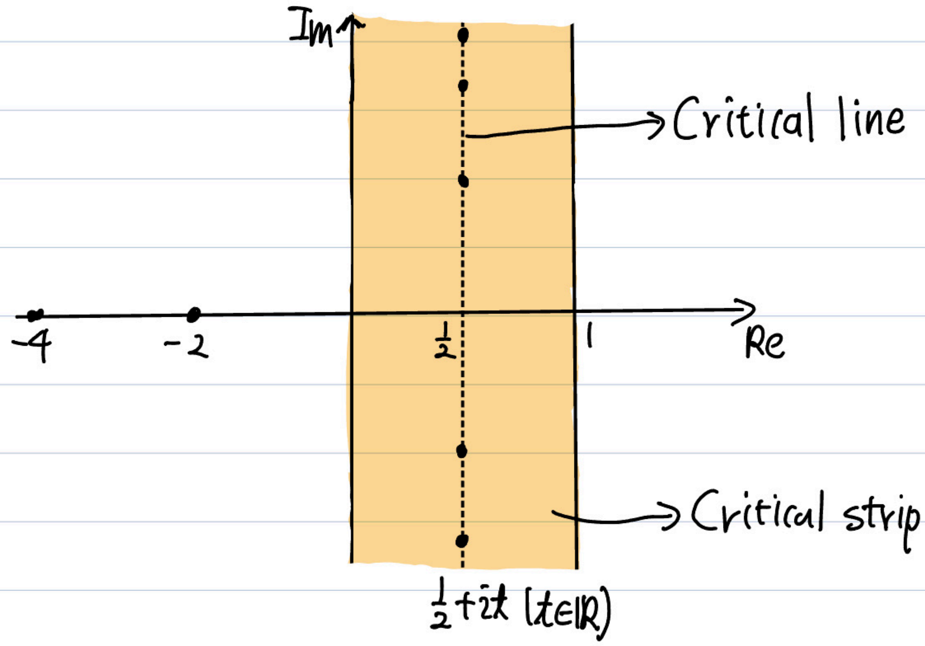
$\therefore$ There is no zero point of ζ at $Re(z)\geq 1$.

By $\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi}{2}s\,\Gamma(1-s)\zeta(1-s)$, There is no zero point of ζ at $Re(z)\leq 1$

What if $0 < Re(s) < 1$?

When $0 < Re(s) < 1$, a number of zero points with a real part $\frac{1}{2}$ are found. However, a zero point with $Re(s) \neq 1/2$ has not yet been found.

Riemann then thought that the real part of zero points of zeta function would be $1/2$ in $0 < Re(s) < 1$.



If $\zeta(\frac{1}{2}+it)=0$, then $\zeta(\frac{1}{2}-it)=0$

$$\zeta(z) = 2^z \pi^{z-1} \sin\frac{\pi}{2}z\, \Gamma(1-z)\zeta(1-z)$$

Thus, if $\zeta(\frac{1}{2}+it)=0$, then $\zeta(1-(\frac{1}{2}+it))=\zeta(\frac{1}{2}-it)=0$