Poincaré Recurrence Theorem

Minsoo Kim*

September 13, 2022

Abstract

Henri Poincaré showed that if the system has a fixed total energy that restricts its dynamics to bounded subsets of its phaase space, the system will eventually return as closely as someone like to any given initial set of particle positions and velocities. However, one can argue that this kind of recurrences are inconsistent with the second law of thermodynamics. Even though the recurrence time is too large to observe, this theoretical discussions enlightened perspective of mankinds about nature. In this seminar, we will discuss how to prove the theorem in a modern way and apply it to the dynamical systems.

1 Systems with One Degree of Freedom

1.1 Definitions

Definition 1.1. A *system with one degree of freedom* is a system described by one differential equation

$$\ddot{x} = f(x) \quad x \in \mathbb{R}. \tag{1}$$

The *kinetic energy* is the quadratic form

$$T = \frac{1}{2}\dot{x}^2.$$

The *potential energy* is the function

$$U(x) = -\int_{x_0}^{x} f(\xi) d\xi.$$

*Department of Physics, Sungkyunkwan University

The *total energy* is the sum

$$E = T + U$$

Notice that the potential energy determines f. Therefore, it is enough to give the potential energy to specify a system of the form (1).

Theorem 1.2. The total energy of points moving according to the equation (1) is conserved: $E(x(t), \dot{x}(t))$ is independent of t.

Proof.

$$\frac{d}{dt}(T + U) = \dot{x}\ddot{x} + \frac{dU}{dx}\dot{x} = \dot{x}(\ddot{x} - f(x)) = 0$$

□

1.2 Phase Flow

Equation (1) is equivalent to the system of two equations:

$$\dot{x} = y \quad \dot{y} = f(x). \tag{2}$$

We consider the plane with coordinates x and y, which we call the *phase plane* of equation (1). The points of the phase plane are called *phase points*. The right-hand side of (2) determines a vector field on the phase space, called the *phase velocity vector field*.

A solution of (2) is a motion $\varphi : \mathbb{R} \to \mathbb{R}^2$ of a phase point in the phase plane, such that the velocity of the moving point at each moment of time is equal to the phase velocity vector at the location of the phase point at that moment.[1]

The image of φ is called that *phase curve*. Thus the phase curve is given by the parametric equations

$$x = \phi(t) \quad y = \dot{\phi}(t).$$

1.3 A Simple Harmonic Oscillator

The basic equation of the theory of oscillations is

$$\ddot{x} = -x.$$

In this case (Figure 1) we have:

$$T = \frac{\dot{x}^2}{2} \quad U = \frac{x^2}{2} \quad E = \frac{\dot{x}^2}{2} + \frac{x^2}{2}.$$

[1] Here we assume for simplicity that the solution φ is defined on the whole time axis $\mathbb{R}$.

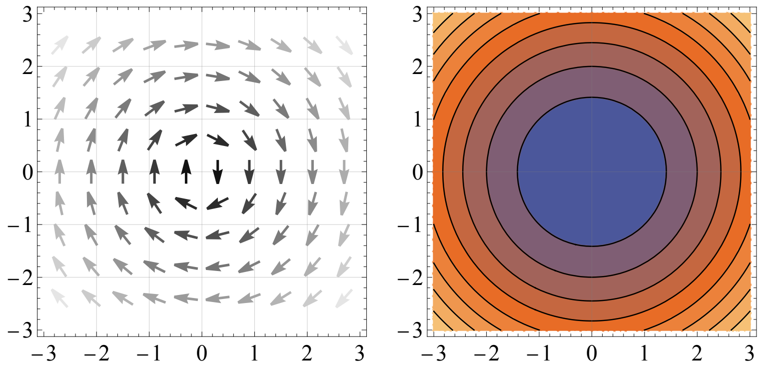

Figure 1: Phase plane of the equation $\ddot{x} = -x$.

The energy level sets are the concentric circles and the origin. The phase velocity vector at the phase point (x, y) has components $(y, -x)$. It is perpendicular to the radius vector and equal to it in magnitude. Therefore, the motion of the phase point in the phase plane is a uniform motion around 0: $x = r_0 \cos(\varphi_0 - t)$, $y = r_0 \sin(\varphi_0 - t)$. Each energy level set is a phase curve.

1.4 Phase Flow

Let M be a point in the phase plane. We look at the solution to system (2) whose initial conditions at $t = 0$ are represented by the point M. We assume that any solution of the system can be extended to the whole time axis. The value of our solution at any value of t depends on M. We denote the resulting phase point by

$$M(t) = g^t M.$$

In this way we have defined a mapping of the phase plane to itself, $g^t : \mathbb{R}^2 \to \mathbb{R}^2$. By theorems in the theory of ordinary differential equations, the mapping g^t is a diffeomorphism (a one-to-one differentiable mapping with a differentiable inverse). The diffeomorphisms g^t, $t \in \mathbb{R}$, form a group: $g^{t+s} = g^t \circ g^s$. The mapping g^0 is the identity ($g^0 M = M$), and g^{-t} is the inverse of g^t. The mapping $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $g(t, M) = g^t M$ is differentiable. All these properties together are expressed by saying that the transformations g^t form a *one-parameter group of diffeomorphisms* of the phase plane. This group is also called the *phase flow*, given by system (2) (or equation (1)). Note that the phase flow given by the equation $\ddot{x} = -x$ is the group g^t of rotations of the phase plane through angle t around the origin.

2 Lagrangian Mechanics

Lagrangian mechanics describes motion in a mechanical system by means of the configuration space. The configuration space of a mechanical system has the structure of a differentiable manifold, on which its group of diffeomorphisms acts. The basic ideas and theorems of lagrangian mechanics are invariant under this group, evne if formulated in terms of local coordinates.

A lagrangian mechanical system is given by a manifold ("configuration space") and a function on its tangent bundle ("the lagrangian function").

Every one-parameter group of diffeomorphisms of configuration space which fixes the lagrangian function defines a conservation law (i.e., a first integral of the equations of motion).

A newtonian potential system is a particular case of lagrangian system (the configuration space in this case is euclidean, and the lagrangian function is the difference between the kinetic and potential energies).

2.1 Calculus of Variations

The calculus of variations is concerned with the extremals of functions whose domain is an infinite-dimensional space: the space of curves. Such functions are called *functionals*.

An example of a functional is the length of a curve in the euclidean plane: if $\gamma = \{(t, x)|x(t) = x, t_0 \leq t \leq t_1\}$, then $\Phi(\gamma) = \int_{t_0}^{t_1} \sqrt{1 + \dot{x}^2}\, dt$.

Definition 2.1. The equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

is called the *Euler-Lagrange equation* for the functional

$$S[x] = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt.$$

Now let x be a vector in the n-dimensional coordinate space $\mathbb{R}^n$, $\gamma = \{(t, x)|x = x(t), t_0 \leq t \leq t_1\}$ a curve in the $(n+1)$-dimensional space $\mathbb{R} \times \mathbb{R}^n$, and $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ a function of $2n + 1$ variables.

Theorem 2.2. The curve γ is an extremal of the functional $S(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt$ on the space of curves joining (t_0, x_0) and (t_1, x_1), if and only if the Euler-Lagrange equation is satisfied along γ.

This is a system of n *second*-order equations, and the solution depends on $2n$ arbitrary constants. The $2n$ conditions $x(t_0) = x_0$, $x(t_1) = x_1$ are used for finding them.

2.2 Lagrange's Equations

Here we indicate the variational principle whose extremals are solutions of Newton's equations of motion in a potential system. We compare Newton's equation of dynamics

$$\frac{d}{dt}(m_i\dot{r}_i) + \frac{\partial U}{\partial r_i} = 0 \tag{3}$$

with the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

Theorem 2.3. Motions of the mechanical system (3) coincide with extermals of the functional

$$S(\gamma) = \int_{t_0}^{t_1} L dt, \quad \text{where} \quad L = T - U$$

is the difference between the kinetic and potential energy.

Proof. Since $U = U(r)$ and $T = \sum_i m_i \|\dot{r}_i\|^2$, we have $\frac{\partial L}{\partial \dot{r}_i} = \frac{\partial T}{\partial \dot{r}_i} = m_i\dot{r}_i$ and $\partial L \partial r_i = -\frac{\partial U}{\partial r_i}$. □

Definition 2.4. In mechanics we use the following terminology: $L(q, \dot{q}, t) = T - U$ is the *Lagrange function* or *lagrangian*, q_i are the *generalized coordinates*, $\dot{q}_i$ are generalized velocities, $\partial L/\partial \dot{q}_i = p_i$ are the *generalized momenta*, $\partial L/\partial q_i$ are *generalized forces*, $\int_{t_0}^{t_1} L(q, \dot{q}, t)dt$ is the *action*, $(d(\partial L/\partial \dot{q}_i)/dt - (\partial L/\partial \dot{q}_i) = 0$ are *Lagrange's equations.*

The last theorem is called "Hamilton's form of the principle of least motion" because in many cases $q(t)$ is not only an extremal but is also a *minimum* value of the cation functional.

3 Hamilton's equations

3.1 Legendre Transformations

The Legendre transformation is a very useful mathematical tool: it transforms functions on a vector space to functions on the dual space. They are often encountered in physics (for example, in the definition of thermodynamic quantities).

Definition 3.1. Suppose the Lagrangian $L : \mathbb{R}^n \to \mathbb{R}$ satisfies these conditions:

1. the mapping $v \mapsto L(v)$ is convex,
2. $\lim_{\|v\|->\infty} \frac{L(v)}{\|v\|} = +\infty$.

The *Legendre transform* of L is

$$L^*(p) = \sup_{q\in\mathbb{R}} \{\langle p, v\rangle - L(v)\} \quad (p \in \mathbb{R}^n).$$

Notice that the convexity implies L is continuous.

For example, let $f(x) = cx^2$ defined on $\mathbb{R}$, where $c > 0$ is a fixed constant. For x^* fixed, the function of x, $x^*x - f(x) = x^*x - cx^2$ has the first derivative $x^* - 2cx$ and second derivative $-2c$; there is one stationary point at $x = x^*/2c$, which is always a minimum. Thus $I^* = \mathbb{R}$ and

$$f^*(x^*) = \frac{x^{*2}}{4c}.$$

One may check the involutivity.

3.2 Hamilton's Equations

We consider the system of Lagrange's equation $\dot{p} = \partial L/\partial q$, where $p = \partial L/\partial \dot{q}$, with a given lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, which we will assume to be convex with respect to the second argument $\dot{q}$.

Theorem 3.2. The system of Lagrange's equations is equivalent to the system of $2n$ first-order equations (Hamilton's equations)

$$\dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{4}$$

where $H(q, p, t) = \langle p, \dot{q}\rangle - L(q, \dot{q}, t)$ is the Legendre transform of the lagrangian function viewed as a function of $\dot{q}$.

Proof. By definition, the Legendre transform of $L(q, \dot{q}, t)$ with respect to $\dot{q}$ is the function $H(q, p, t) = \langle p, \dot{q}\rangle - L(q, \dot{q}, t)$, in which $\dot{q}$ is expressed in terms of p by the formula $p = \partial L/\partial \dot{q}$, and which depends on the parameters q and t. This function H is called the *hamiltonian*.

The total differential of the hamiltonian

$$dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt$$

is equal to the total differential of $p \cdot \dot{q} - L$ for $p = \partial L/\partial \dot{q}$:

$$dH = \dot{q} \cdot dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt.$$

Both expression for dH must be the same. Therefore,

$$\dot{q} = \frac{\partial H}{\partial p} \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Applying Lagrange's equations $\dot{p} = \partial L / \partial q$, we obtain Hamilton's equations. □

Notice that the converse is proved in an analogous manner. Therefore, the systems of Lagrange and Hamilton are equivalent.

Theorem 3.3. Under the following assumptions, the hamiltonian H is the total energy $H = T + U$.

$$T = \frac{1}{2} \sum_{ij} a_i j \dot{q}_i \dot{q}_j$$

where $a_{ij} = a_{ij}(q,t)$ and $U = U(q)$.

3.3 Liouville's Theorem

For simplicity we look at the case in which the hamiltonian function does not depend explicitly on the time: $H = H(q,p)$.

Definition 3.4. The $2n$-dimensional space with coordinates $q_1, \ldots, q_n; p_1, \ldots, p_n$ is called phase space.

The right-hand sides of Hamilton's equations give a vector field: at each point (q,p) of phase space there is a $2n$-dimensional vector $(\partial H/\partial p, -\partial H/\partial q)$. We assume that every solution of Hamilton's equations can be extended to the whole time axis.[2]

Definition 3.5. The *phase flow* is the one-parameter group of transforamtions of phase space

$$g^t : (q(0), p(0)) \mapsto (q(t), p(t)),$$

where $q(t)$ and $p(t)$ are solutions of Hamilton's system of equations (Figure 2).

Notice that $\{g^t\}$ is a group.

[2]For this it is sufficient, for example, that the level sets of H be compact.

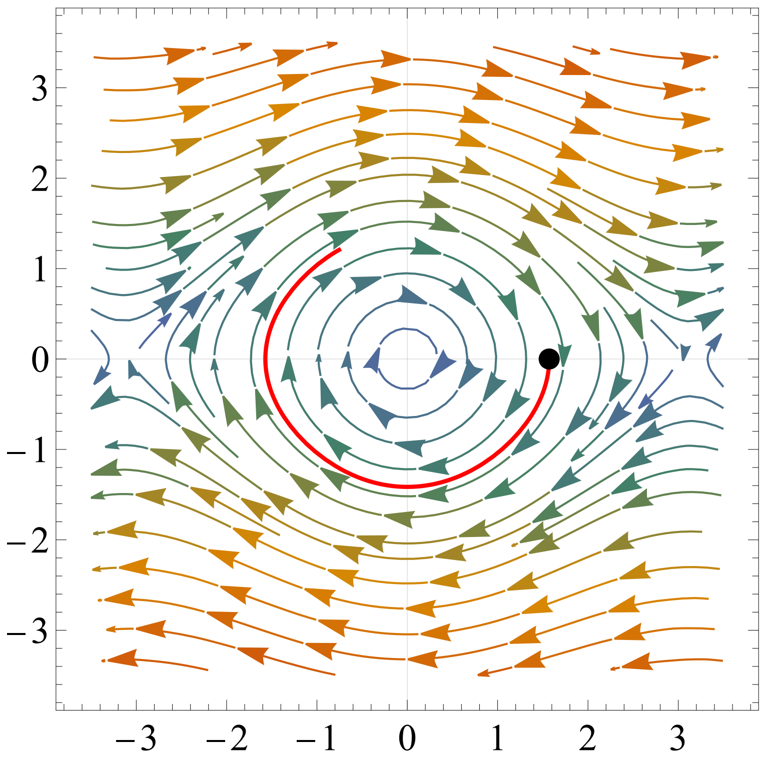

Figure 2: Phase flow of a simple pendulum. The black point is an initial state. It moves along the red curve.

Suppose we are given a system of ordinary differential equations $\dot{x} = f(x)$, $x = (x_1, \ldots, x_n)$, whose solution may be extended to the whole time axis. Let $\{g^t\}$ be the corresponding group of transformations:

$$g^t(x) = x + f(x)t + O(t^2). \tag{5}$$

Let $D(0)$ be a region in x-space and $v(0)$ its volume;

$$v(t) = \mathrm{vol}(D(t)) \quad D(t) = g^t D(0).$$

Lemma 3.6. For any matrix $A = (a_{ij})$,

$$\det(I_n + At) = 1 + t\,\mathrm{tr}(A) + O(t^2),$$

where $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$ is the trace of A.

Lemma 3.7. $(dv/dt)|_{t=0} = \int_{D(0)} \mathrm{div}(f)\,dx$, $(dx = dx_1 \ldots dx_n)$.

Proof. For any t, the formula for changing variables in a multiple integral gives

$$v(t) = \int_{D(0)} \det \frac{\partial g^t x}{\partial x} dx.$$

Calculating $\partial g^t x / \partial x$ by formula (5), we find

$$\frac{\partial g^t x}{\partial x} = I_n + \frac{\partial f}{\partial x} + O(t^2).$$

Using lemma 3.6, we have

$$\det \frac{\partial g^t x}{\partial x} = 1 + t \operatorname{tr} \frac{\partial f}{\partial x} + O(t^2).$$

But $\operatorname{tr} \partial f / \partial x = \sum_{i=1}^n \partial f_i / \partial x_i = \operatorname{div}(f)$. Therefore,

$$v(t) = \int_{D(0)} \left(1 + t \operatorname{div}(f) + O(t^2)\right) dx,$$

which prove the statement. □

Lemma 3.8. If $\operatorname{div}(f) = 0$, then g^t preserves volume: $v(t) = v(0)$.

Proof. Without loss of generality, $t = t_0$ is no worse than $t = 0$. Then lemma 3.7 can be written in the form

$$\left. \frac{dv(t)}{dt} \right|_{t=t_0} = \int_{D(t_0)} \operatorname{div}(f) dx,$$

and if $\operatorname{div}(f) = 0$, $dv/dt = 0$. □

Theorem 3.9. The phase flow preserves volume: for any region D we have

$$\operatorname{vol}(g^t D) = \operatorname{vol}(D).$$

Proof. For Hamilton's equation (4), we have

$$\operatorname{div}(f) = \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0.$$

This proves Liouville's theorem. □

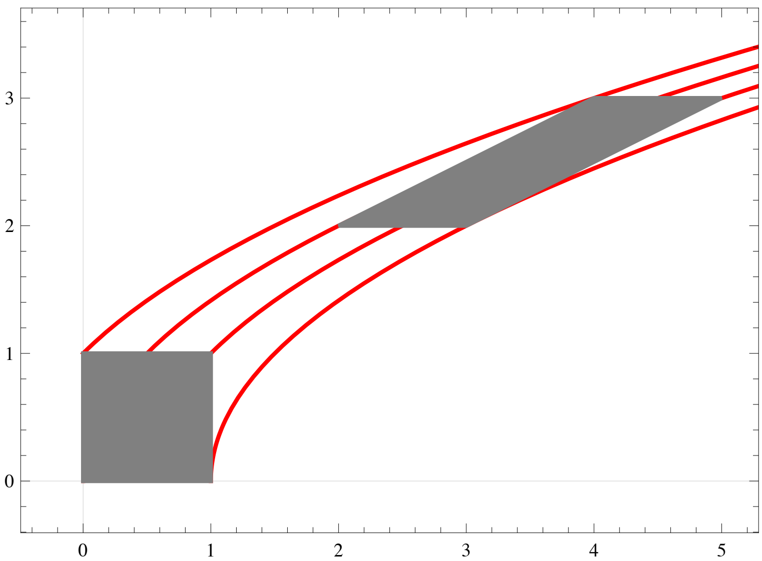

Figure 3: Four phase space trajectories for falling objects.

4 Poincaré's Recurrence Theorem

Theorem 4.1. Let g be a volume-preserving continuous one-to-one mapping which maps a bounded region D of euclidean space onto itself: $gD = D$. Then in any neighborhood U of any point of D there is a point $x \in U$ which returns to U, i.e., $g^n x \in U$ for some $n > 0$.

Proof. We consider the images of the neighborhood U:

$$U, gU, g^2U, \ldots, g^nU, \ldots$$

All of these have the same volume. If they never intersected, D would have infinite volume. Therefore, for some $k \geq 0$ and $l \geq 0$, with $k > l$,

$$g^kU \cap g^lU \neq \varnothing.$$

Therefore, $g^{k-l}U \cap U \neq \varnothing$. If y is in this intersection, then $y = g^{k-l}x$, with $x \in U$. Now let $n = k - l$, then $x \in U$ and $g^n x \in U$. $\square$

Poincaré recurrence theorem shows that almost every moving point returns repeatedly to the vicinity of its initial position.