Time asymptotic behavior of compressible Euler equation

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Compressible Euler equation

$$\begin{cases} \partial_t \rho + div_x(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla_x u + \nabla_x p = 0, \end{cases} \qquad \cdots \ (OE)$$

where $\rho(t,x)$: density at time t, position x

$u(t,x)$: velocity

$p(t,x)$: pressure.

$\rho,\ p : \mathbb{R}^d \longrightarrow \mathbb{R}$ and $u : \mathbb{R}^d \longrightarrow \mathbb{R}^d$.

In d-dimension, the above equation consists of $(d+1)$ scalar equations, and $(d+2)$ unknowns.
So, underdetermined.

Formal derivation

Assume : Solutions are C^∞.

(1) Conservation law of mass under continuum assumption

: Let Ω be a open set with smooth boundary in $\mathbb{R}^3$.

Note that

$$\frac{d}{dt}\int_\Omega \rho \, dV = -\int_{\partial\Omega} \rho u \cdot \nu \, dS$$

where ν is the outward normal vector field of $\partial\Omega$.

The left hand side means 'mass of fluid in Ω'.

The right hand side means 'flow of fluid in Ω'.

By Divergence Theorem,

RHS $= - \int_{\partial\Omega} div_x(\rho u)\, dV$.

$\therefore \int_{\partial\Omega} \partial_t \rho + div_x(\rho u)\, dV = 0$

Since Ω is arbitrary smooth domain, this concludes

$$\partial_t \rho + div_x(\rho u) = 0.$$

$\square$

(2) Conservation law of momentum

: Let $x(t) = (x_1(t), x_2(t), x_3(t))$ be the "particle trajectory".
(i.e., the path followed by a fluid particle)

defined by $x'(t) = u(t, x(t))$.

Then the acceleration equals $x''(t) = \partial_t u + u \cdot \nabla_x u$.

Define the linear operator $\frac{D}{Dt} := \partial_t + u \cdot \nabla_x$.

(In continuum mechanics, the material derivative describes the time rate of change of some physical quantity (like heat or momentum) of a material element that is subjected to a space-and-time-dependent macroscopic velocity field.)

By Newton's second law,

$$\text{"force per unit volume} = \rho \frac{Du}{Dt}.\text{"}$$

We consider two types of forces : Surface force and Body force.

For an ideal fluid, there exists pressure $p(t, x)$ such that

$$\text{"force acting on } \Omega = -\int_{\partial\Omega} p\nu \, dS.\text{"}$$

For the given body force f per unit mass,

$$\text{"the body force acting on } \Omega = \int_{\Omega} \rho f \, dV.\text{"}$$

In the same way above, applying Divergence Theorem, we get

$$\rho \frac{Du}{Dt} + \nabla_x p = \rho f. \qquad \square$$

Remark1)

$$\begin{cases} f\equiv 0 \implies (OE)_2 \text{ is called "conservation law of momentum"}, \\ f\not\equiv 0 \implies (OE)_2 \text{ is called "balance law of momentum"}, \end{cases}$$

Remark2)

$$(OE)_2 \iff \partial_t(\rho u) + div_x(\rho u \otimes u) + \nabla_x p = 0.$$

$$\begin{aligned} \because \quad & j\text{-th component of } \partial_t(\rho u) + div_x(\rho u \otimes u) \\ &= \partial_t(\rho u_i) + \partial_{x_j}(\rho u_i u_j) \\ &= u_i\partial_t\rho + \rho\partial_t u_i + u_i\partial_{x_j}(\rho u_j) + \rho u_j\partial_{x_j} u_i \\ &= u_i(\partial_t\rho + \partial_{x_j}(\rho u_j)) + \rho(\partial_t u_i + u_j\partial_{x_j} u_i) \\ &= \rho(\partial_t u_i + u_j\partial_{x_j} u_i) \text{ (Apply (OE)}_1.) \\ &= \rho(\partial_t u_i + u\cdot\nabla_x u_i). \end{aligned}$$

□

After physical correction and supplementation, (OE) changes. We call

(1) Incompressible Euler equation :

$$\begin{cases} \partial_t \rho + div_x(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla_x u + \nabla_x p = 0, \\ divu = 0. \end{cases}$$

(2) Compressible Euler equation :

$$\begin{cases} \partial_t \rho + div_x(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla_x u + \nabla_x p = 0, \\ \left(\rho\left(\frac{|u|^2}{2} + e\right)\right)_t + div_x\left(\left(\rho\left(\frac{|u|^2+e}{2}\right)\right)u\right) = 0. \end{cases}$$

where $e(t,x)$ denotes a internal energy.

Note that if last equation of (2) is deleted, then we call 'isentropic(barotropic) C.E'.

Def 1) 1-dimensional Compressible Barotropic Euler equation

$$\begin{cases} v_t - u_x = 0, & x \in \mathbb{R}, t \geq 0 \\ u_t + p(v)_x = 0, & \end{cases} \qquad \cdots \ (b)$$

with the Riemann initial data

$$(\mathrm{v}(0,\, \mathrm{x}),\ \mathrm{u}(0,\, \mathrm{x})) = \begin{cases} (v_-, u_-), & x < 0 \\ (v_+, u_+), & x > 0. \end{cases}$$

Def 2) 1-dimensional Barotropic Navier Stokes Equation

$$\begin{cases} v_t - u_x = 0, & x \in \mathbb{R}, t \geq 0 \\ u_t + p(v)_x = (\frac{\mu u_x}{v})_x, & \end{cases} \qquad \cdots (a)$$

where $b > 0$, $\gamma > 1$ and $\mu > 0$ are constants and $p(v) = bv^{-\gamma}$.

We set $\mu = 1$ and $b = 1$.

This system is then endowed with initial values:

$$\begin{cases} (v(0,x), u(0,x)) = (v_0(x), u_0(x)), & x \in \mathbb{R} \\ (v_0(x), u_0(x)) \longrightarrow (v_\pm, u_\pm), & as\ x \longrightarrow \pm\infty \end{cases}$$

We consider the solutions of above two equations

"in the weak sense",

for example, (v, u) satisfies (b) if and only if

$$-\int_{-\infty}^{+\infty}\int_{0}^{+\infty} v\frac{\partial \phi}{\partial t} - u\frac{\partial \phi}{\partial x} dtdx = 0 \text{ and}$$

$$-\int_{-\infty}^{+\infty}\int_{0}^{+\infty} u\frac{\partial \phi}{\partial t} + p(v)\frac{\partial \phi}{\partial x} dtdx = 0,$$

for all $\phi \in C_c(\mathbb{R}^+ \times \mathbb{R})$.

Def3) Viscous Shock Wave

Let $\sigma := \sqrt{-\frac{(p(v_+)-p(v_m))}{(v_+-v_m)}}$.

From (a), consider the below ODE.

$$\begin{cases} -\sigma(\tilde{v}^S)' - (\tilde{u}^S)' = 0, \\ -\sigma(\tilde{u}^S)' + (p(\tilde{v}^S))' = (\frac{\tilde{u}^S}{\tilde{v}^S})' \\ (\tilde{v}^S, \tilde{u}^S)(-\infty) = (v_-, u_-), \quad (\tilde{v}^S, \tilde{u}^S)(+\infty) = (v_+, u_+) \end{cases}$$

For this solution $\tilde{v}^S$ and $\tilde{u}^S$, $(\tilde{v}^S(x-\sigma t), \tilde{u}^S(x-\sigma t))$ is the solution is a solution of (a).

This is called Viscous shock wave of (a).

The space which the above ODE has solution is $S(v_+, u_+)$.
If $(v_-, u_-) \in S(v_+, u_+)$, then there exists viscous shock wave.

Def4) Rarefaction Wave

Note that $(b) \iff \mathbf{v} + A(\mathbf{v})\mathbf{v}_x = 0$,
where $\mathbf{v} = (v, u)$ and $A(\mathbf{v}) = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$.

The eigen values of $A(\mathbf{v})$ are $\lambda_1(v) = -\sqrt{-p'(v)}$
and $\lambda_2(v) = \sqrt{-p'(v)}$.

At this time, the 1-rarefaction curve of (b) is defined by

$$R_1(v_+, u_+) := \left\{ (v, u) \in \mathbb{R}^2 : v < v_+, \quad u = u_+ - \int_{v_+}^{v} \lambda_1(s) ds \right\}.$$

In addtion, the 1-rarefaction wave (v_r, u_r) of (b) is defined by

$$\lambda_1(v^r(t,x)) := \begin{cases} \lambda_1(v_-), & x < \lambda_1(v_-)t \\ \frac{x}{t}, & \lambda_1(v_-)t \le x \le \lambda_1(v_+)t \\ \lambda_1(v_+), & x > \lambda_1(v_+)t \end{cases},$$

together with $z_1(v^r(t,x), u^r(t,x)) = z_1(v_-, u_-) = z_1(v_+, u_+)$, where $z_1(v,u) = u + \int_{v_+}^{v} \lambda_1(s)ds$.
This is a solution of (b).

The 2-rarefaction curve and wave are treated similarly.

Def5) Composite wave of viscous shock and rarefaction.

Given the end states $(v_\pm, u_\pm) \in \mathbb{R}^+ \times \mathbb{R}$ in (a),

we consider the case that there exists a unique intermediate state

(v_m, u_m) such that $(v_-, u_-) \in R_1(v_m, u_m)$, $(v_m, u_m) \in S_2(v_+, u_+)$.

We will consider a superposition wave:

$$(v^r(\tfrac{x}{t}) + \tilde{v}^S(x - \sigma t) - v_m, u^r(\tfrac{x}{t}) + \tilde{u}^S(x - \sigma t) - u_m).$$

Def6) Smooth approximation rarefaction wave

Consider the below Burgers equation :

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = \frac{w_m + w_-}{2} + \frac{w_m - w_-}{2}\tanh x \end{cases}$$

In the similar way of 'Def4)', we get the 'approximate rarefaction wave' $(\tilde{u}^R(t,x), \tilde{v}^R(t,x))$.
In addition, this wave satisfies the system:

$$\begin{cases} \tilde{v}^R_t - \tilde{u}^R_x = 0 \\ \tilde{u}^R_t + p(\tilde{v}^R)_x = 0 \end{cases}$$

The reason considering Smooth approximation rarefaction wave is the solution is good property and it can be approximated to original rarefaction wave.

Main Theorem

For a given constant state $(v_+, u_+) \in \mathbb{R}_+ \times \mathbb{R}$, there exists constants δ_0, $\epsilon_0 > 0$ such that the followings holds true.
For any $(v_m, u_m) \in S_2(v_+, u_+)$ and $(v_-, u_-) \in R_1(v_m, u_m)$ such that $|v_+ - v_m| + |v_m - v_-| \leq \delta_0$,
denote $(v^r, u^r)(\frac{x}{t})$ the 1-rarefaction solution to (b) with end sates (v_-, u_-) and (v_m, u_m), and $(\tilde{v}^S, \tilde{u}^S)(x - \sigma t)$ the 2-viscous shock solution of (a) with end states (v_m, u_m) and (v_+, u_+). Let (v_0, u_0) be any initial data such that

$$\sum_{\pm}(||v_0 - v_\pm||_{L^2(\mathbb{R}_\pm)} + ||u_0 - u_\pm||_{L^2(\mathbb{R}_\pm)}) \\ +||v_{0x}||_{L^2(\mathbb{R})} + ||u_{0x}||_{L^2(\mathbb{R})} < \varepsilon_0,$$

where $\mathbb{R}_- := -\mathbb{R}_+ = (-\infty, 0)$.
Then the compressible Navier-Stokes system (a) admits a unique global-in-time solution (v, u). Moreover, there exists an absolutely continuous shift $X(t)$ such that

$$v(t,x)-\left(v^r(\tfrac{x}{t})+\tilde{v}^S(x-\sigma t-X(t))-v_m\right)\in C(0,+\infty;H^1(\mathbb{R}),$$

$$u(t,x)-\left(u^r(\tfrac{x}{t})+\tilde{u}^S(x-\sigma t-X(t))-u_m\right)\in C(0,+\infty;H^1(\mathbb{R}),$$

$$u_{xx}(t,x)-\tilde{u}^S_{xx}(x-\sigma t-X(t))\in L^2(0,\infty;L^2(\mathbb{R}).$$

In addition, as $t\longrightarrow+\infty$,

$$\sup_{x\in\mathbb{R}}|(v,u)(t,x)-(v^r(\tfrac{x}{t})+\tilde{v}^S(x-\sigma t-X(t))-v_m,\\ \mathrm{u}^r(\tfrac{x}{t})+\tilde{u}^S(x-\sigma t-X(t))-u_m)|\longrightarrow 0,$$

and

$$\lim_{t\to\infty}|X'(t)|=0.$$

For simplification of our analysis, we rewrite the compressible Navier-Stokes system (1.1) into the following system,

$$\begin{cases} v_t - \sigma v_\xi - u_\xi = 0, \\ u_t - \sigma u_\xi + p(v)_\xi = (\frac{u_\xi}{v})_\xi. \end{cases} \quad \cdots \text{ (c)}$$

That is $(t, x) \longmapsto (t, \xi = x - \sigma t)$.

Moreover, for the more simpler argument, we use a clever substitution

$$h := u - (lnv)_\xi.$$

That is the coordinate (v, u) change into (v, h).
Then the above equation is transformed into

$$\begin{cases} v_t - \sigma v_\xi - h_\xi = (lnv)_{\xi\xi}, \\ h_t - \sigma u_\xi + p(v)_\xi = 0. \end{cases} \quad \cdots \text{ (d)}$$

Def7) Weight function and Shift function

1) We define the weight function a by

$$a(\xi) := 1 + \frac{\lambda}{\delta_S}(p(v_m) - p(\tilde{v}^S(\xi)),$$

where the constant λ is chosen to be so small but far bigger than δ_S such that $\delta_S << \lambda \leq C\sqrt{\delta_S}$.
Note that
$1 < a(\xi) < 1 + \lambda$, $a'(\xi) = -\frac{\lambda}{\delta_S} p'(\tilde{v}^S)\tilde{v}_\xi{}^S > 0$, and $|a'| \sim \frac{\lambda}{\delta_S}\left|\tilde{v}_\xi^S\right|$.

2) We define the shift X as a solution to the ODE:

$$\begin{cases} X'(t) = -\frac{M}{\delta_S}\left[\int_{\mathbb{R}} \frac{a(\xi - X)}{\sigma} \partial_\xi \tilde{h}^S(\xi - X)(p(v) - p(\tilde{v}_X))d\xi \right. \\ \qquad\qquad \left. - \int_{\mathbb{R}} a(\xi - X)\partial_\xi p(\tilde{v}^S(\xi - X))(v - \tilde{v}_{-X})d\xi\right], \\ X(0) = 0, \end{cases}$$

where $M := \frac{5(\gamma+1)\sigma_m^3}{8\gamma p(v_m)}$ with $\sigma_m := \sqrt{-p'(v_m)}$.

Proposition

Define $(\tilde{v}_{-X}, \tilde{u}_{-X})(t,\xi) :=$
$(\tilde{v}^R(t,\xi+\sigma t)+\tilde{v}^S(\xi-X(t))-v_m, \tilde{u}^R(t,\xi+\sigma t)+\tilde{u}^S(\xi-X(t))-u_m)$.

For a given point $(v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$, there exist positive constants $C_0, \delta_0, \epsilon_1$ such that the following holds.
Suppose that (v, u) is the solution to to (c) on $[0, T]$ for some $T > 0$. Assume that both the rarefaction and shock waves strength satisfy δ_R, $\delta_S < \delta_0$ and that
$v - \tilde{v}_{-X} \in C([0, T]; H^1(\mathbb{R})$,
$u - \tilde{u}_{-X} \in C([0, T]; H^1(\mathbb{R}) \cap L^2(0, T; H^2(\mathbb{R}))$
and
$||v - \tilde{v}_{-X}||_{L^\infty(0,T;H^1(\mathbb{R}))} + ||u - \tilde{u}_{-X}||_{L^\infty(0,T;H^1(\mathbb{R}))} \leq \varepsilon_1$.

Then for all $t \leq T$,

$$sup_{t\in[0,T]}\left[||v-\tilde{v}_{-X}||_{H^1(\mathbb{R})}+||u-\tilde{u}_{-X}||_{H^1(\mathbb{R})}\right]+$$
$$\sqrt{\delta_S\int_0^t |X'(s)|^2\,ds}+\sqrt{\int_0^t G^S(U)+G^R(U)+D(U)+D_1(U)+D_2(U)ds}$$
$$\leq C_0(||v_0-\tilde{v}(0,\xi)||_{H^1(\mathbb{R})}+||u_0-\tilde{u}(0,\xi)||_{H^1(\mathbb{R})}+\delta_R^{\frac{1}{6}}),$$

where C_0 is independent of T and

$$G^S(U):=\int_{\mathbb{R}}\left|\tilde{v}^S_\xi(\xi-X(t))\right||v-\tilde{v}_{-X}|^2\,d\xi,$$
$$G^R(U):=\int_{\mathbb{R}}\left|\tilde{u}^R_\xi\right||v-\tilde{v}_{-X}|^2\,d\xi,$$
$$D(U):=\int_{\mathbb{R}}|\partial_\xi(p(v)-p(\tilde{v}_{-X})|^2\,d\xi,$$
$$D_1(U):=\int_{\mathbb{R}}|(u-\tilde{u}_{-X})_\xi|^2\,d\xi$$
$$D_2(U):=\int_{\mathbb{R}}|(u-\tilde{u}_{-X})_{\xi\xi}|^2\,d\xi.$$

In addition,

$$|X'(t)|\leq C_0\,|(v-\tilde{v}_{-X})(t,\xi)|_{L^\infty(\mathbb{R})},\qquad t\leq T.$$

Bootstrap Argument

(i) Prove that the existence of solution satisfying some inequality in some time intervals.

(ii) Suppose the maximal time which is able to exist solution is finite.

(iii) Prove that we can apply (i) inductively changing the initial time.

(iv) Show the contradiction of (ii).

Relative Entropy Method

Define $Q(v) := \frac{1}{(\gamma-1)v^{\gamma-1}}$,

$$Q(v|\tilde{v}) := Q(v) - Q(\tilde{v}) - (v - \tilde{v})(Q'(\tilde{v})), \text{ and}$$

$$\eta(U|\tilde{U}) := \frac{|h-\tilde{h}|^2}{2} + Q(v|\tilde{v}), \text{ where } U = (v, h).$$

Then we enough to show that

$$\frac{d}{dt}\int_{\mathbb{R}} a^{-X}(\xi)\eta(U(t,\xi)|\tilde{U}(t,\xi))d\xi \leq 0.$$

In fact,

$$\frac{d}{dt}\int_{\mathbb{R}} a^{-X}(\xi)\eta(U(t,\xi)|\tilde{U}(t,\xi))d\xi$$

$$= X'(t)Y(U) + \mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U).$$

At this time, all terms of $\mathcal{J}^{good}(U)$ have positive sign.

Kang - Vasseur inequality

For any $f : [0,1] \longrightarrow \mathbb{R}$ satisfying $\int_0^1 y(1-y)\,|f'|^2\,dy < \infty$,

$$\int_0^1 \left| f - \int_0^1 f dy \right|^2 dy \leq \frac{1}{2} \int_0^1 y(1-y)\,|f'|^2\,dy.$$

This inequality is sharp.

In the process of calculation, we substitute $y = \frac{p(v_m)-p(\tilde{v}^S)}{\delta_S}$.
At this time, this inequality give a critical bound.

Improved proof performed by Lee Ho Bin and Han Sung Ho

Using the substitution

$$'h = u - (lnv)_\xi'$$

has advantages giving simple calculus and easy control the solutions.

However, this method is applicable only to one-dimensional case.

Since Navier-Stokes and Euler equation are not just the form of 1-dimensional, for more general case study we have to use the general method or tool.

Thus we attempt the proof without above substitution.
This process is harder than using substitution.
The idea of proof is somewhat similar, but the more calculus and the more idea to control solutions are required.

Thank You!