History of Mathematics

A Journey to Find a Foundation of Mathematics

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Timeline

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1650 BC, Egypt

Rhind Mathematical Papyrus

#50 Question

Find the area of a circle with a diameter of 9

1650 BC, Egypt

Rhind Mathematical Papyrus

#50 Question

Find the area of a circle with a diameter of 9

Recipe

To find the area of the circle, subtract 1/9 of the diameter and square it.

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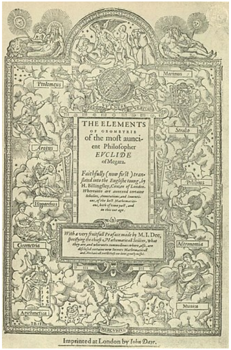
400 BC∼ 300 BC



≪ *Element* ≫

≪ *Element* ≫

- Definition
- Postulate
- Axiom

23 Definitions

1. A point is that which has no part
2. A line is breadthless length
3. The extremities of a line are points
4. A straight line is a line which lies evenly with the points on itself
5. A surface is that which has length and breath only
6. The extremities of a surface are lines
7. A plane surface is a surface which lies evenly with the straight lines on itself

$\vdots$

Euclid, The Beginning of Mathematics

Postulates

1. Can draw a straight-line from any point to any point
2. Can produce a finite straight-line continuously in a straight-line
3. Can draw a circle with any center and radius
4. All right-angles are equal to one another
5. The sum of the inner angles of a triangle is 180 degrees

Axioms

1. $A = B \,\&\, A = C \Rightarrow B = C$
2. $A = B \Rightarrow A + C = B + C$
3. $A = B \Rightarrow A - C = B - C$
4. Things coinciding with one another are equal to one another
5. The whole is greater than the part

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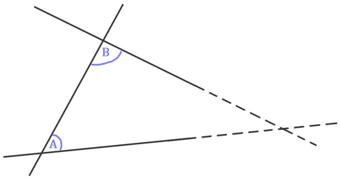
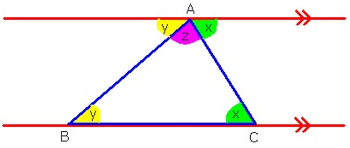
Fifth Postulate, Parallel Postulate

Parallel Postulate

- If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles
- The sum of the inner angles of a triangle is 180 degrees





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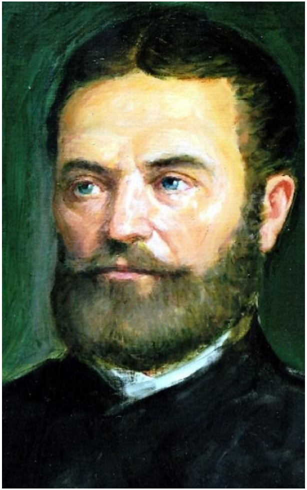
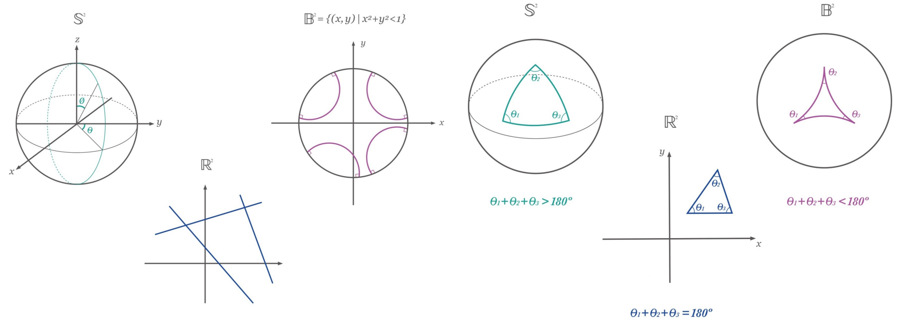
1823



Bolyai János

Non-Euclidean Geometry



Spaces

- Euclidean Geometry
- Hyperbolic Geometry
- Spherical Geometry

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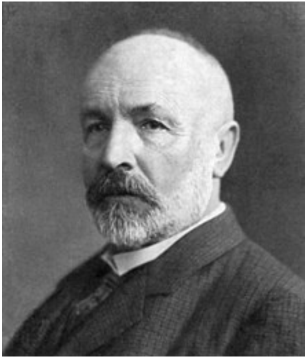

Georg Cantor(1845$\sim$ 1918)

$$\mathbb{N} = \{0, 1, 2, 3, \cdots\}$$

Cantor, Set Theory

Peano's axioms

1. 0 is a natural number
2. For every natural number x, $x = x$
3. For all natural numbers x and y, if $x = y$, then $y = x$
4. For all natural numbers x, y and z, if $x = y$ and $y = z$, then $x = z$
5. For all a and b, if b is a natural number and $a = b$, then a is also a natural number
6. For every natural number n, $S(n)$ is a natural number (S is a successor function)
7. For all natural numbers m and n, if $S(m) = S(n)$, then $m = n$
8. For every natural number n, $S(n) = 0$ is false.

What is "counting"?

$\leq$

Let A, B be sets. Let $\#(\cdot)$ be a cardinality function. Then

$$\#(A) \leq \#(B) \Leftrightarrow \exists \text{ injection map } : A \to B$$

Cantor, Set Theory

$\leq$

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$=$

Let A, B be sets. Let $\#(\cdot)$ be a cardinality function. Then

$$\#(A) = \#(B) \Leftrightarrow \exists \text{ bijection map } : A \to B$$

$$\#\mathbb{N} \;\; <?, =?, >? \; \#\mathbb{Z}$$

$\#\mathbb{N} = \#\mathbb{Z}$

$\#\mathbb{N} = \#\mathbb{Q}$

$\#\mathbb{N} \lneq \#\mathbb{R}$

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The Second Crisis of Mathematics, Paradoxes

Cantor's Theorem

Let A be a set. Then

$$\#A \lneq \#2^A$$

where 2^A is a power set of A

Cantor's Paradox

Let A be a set of all sets. By Cantor's theorem

$$\#A \lneq \#2^A$$

But $2^A \subset A$. $\therefore \#A \geq \#2^A \quad \rightarrow\leftarrow$

1901

Russell's Paradox

Let $A = \{x | x \notin x\}$ Then

$$A \in A \text{ or } A \notin A$$

$(A \in A)$
Assume that $A \in A$. But A defined as $A = \{x | x \notin x\}$
$\therefore A \notin A$

$(A \notin A)$
Assume that $A \notin A$. But A defined as $A = \{x | x \notin x\}$
$\therefore A \in A$

$\rightarrow\leftarrow$

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New Foundation, ZFC

Zermelo-Fraenkel,ZF

- Axiom of Extensionality
- Axiom of Existence
- Axiom schema of Specification
- Axiom of Pairing
- Axiom of The union
- Axiom of The power set
- Axiom of Infinity
- Axiom of Foundation
- Axiom schema of Replacement

New Foundation, ZFC

Choice function

Let X be a set. Let $P'(X) := 2^X - \emptyset$
Let $r : P'(X) \to X$ is called a **choice function** for X if
$\forall B \in P'(X), r(B) \in B$

New Foundation, ZFC

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Axiom of Choice

Every set has a choice function

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David Hilbert(1862∼1943)

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Gödel's Incompleteness Theorems

Kurt Gödel(1906~1978)

Gödel's Incompleteness Theorems

Gödel's First Incompleteness Theorems

Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; *i.e.*, there are statements of the language of F which can neither be proved nor disproved in F

Gödel's Second Incompleteness Theorems

For any consistent system F within which a certain amount of elementary arithmetic can be carried out, the consistency of F cannot be proved in F itself

Timeline

Das Wesen der Mathematik liegt in ihrer Freiheit

-Georg Ferdinand Ludwig Philipp Cantor