Jones Polynomial

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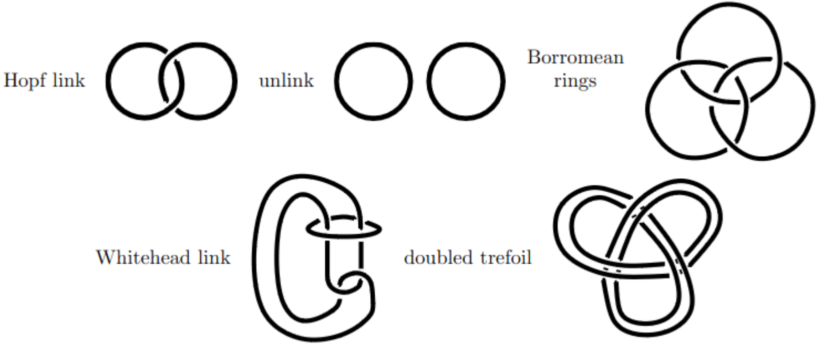
Knot and Link

Link

A link is simply a collection of (finitely-many) disjoint closed loops of string in $\mathbb{R}^3$; each loop is called a component of the link.



Knot

A knot is a closed loop of string in $\mathbb{R}^3$; two knots are equivalent if one can be wiggled around, stretched, tangled and untangled until it coincides with the other. Cutting and rejoining is not allowed.

Knot


0_1 3_1 4_1 5_1 5_2 6_1 6_2 6_3 7_1
7_2 7_3 7_4 7_5 7_6 7_7 8_1 8_2 8_3
8_4 8_5 8_6 8_7 8_8 8_9 8_{10} 8_{11} 8_{12}
8_{13} 8_{14} 8_{15} 8_{16} 8_{17} 8_{18} 8_{19} 8_{20} 8_{21}
9_1 9_2 9_3 9_4 9_5 9_6 9_7 9_8 9_9
9_{10} 9_{11} 9_{12} 9_{13} 9_{14} 9_{15} 9_{16} 9_{17} 9_{18}
9_{19} 9_{20} 9_{21} 9_{22} 9_{23} 9_{24} 9_{25} 9_{26} 9_{27}
9_{28} 9_{29} 9_{30} 9_{31} 9_{32} 9_{33} 9_{34} 9_{35} 9_{36}
0^2_1 2^2_1 4^2_1 5^2_1 6^2_1 6^2_2 6^2_3 7^2_1 7^2_2
7^2_3 7^2_4 7^2_5 7^2_6 7^2_7 7^2_8 8^2_1 8^2_2 8^2_3
8^2_4 8^2_5 8^2_6 8^2_7 8^2_8 8^2_9 8^2_{10} 8^2_{11} 0^3_1
6^3_1 6^3_2 6^3_3 7^3_1 8^3_1 8^3_2 8^3_3 8^3_4 8^3_5

Isotopic

Isotopic

Isotopy is a homotopy H such that for each fixed t, $H(x,t)$ gives an embedding.

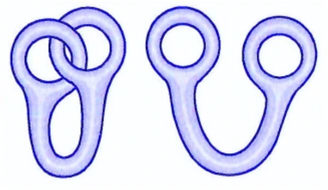


Figure: Isotopic

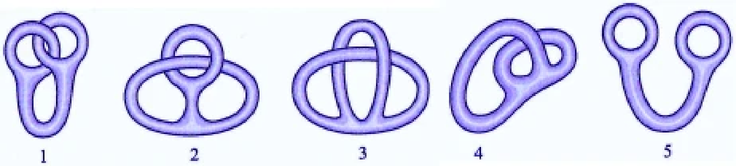
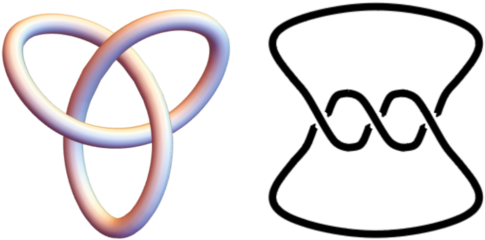


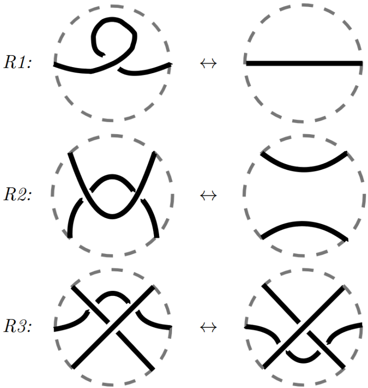
Figure: Isotopic

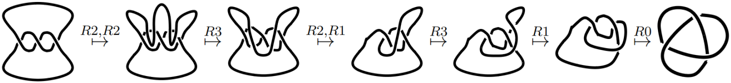
Diagram

We have many different diagrams of the same knot!
Example)

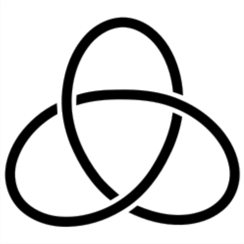


Reidemeister moves

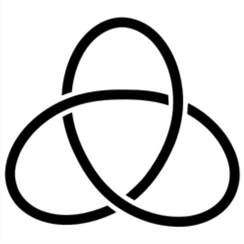



R2,R2
R3
R2,R1
R3
R1
R0

Mirror-image



(a) Left Trefoil



(b) Right Trefiol

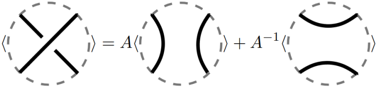
Figure: Mirror images of Trefoils

Kauffman bracket

Definition

The Kauffman bracket polynomial of an unoriented link diagram D is a Laurent polynomial $\langle D \rangle \in \mathbb{Z}\left[A^{\pm 1}\right]$, defined by the following recursive rules.

1. It is invariant under planar isotopy of diagrams.
2. Skein relation



3. $\langle D \sqcup U \rangle = \left(-A^2 - A^{-2}\right) \langle D \rangle$ where U is any closed crossingless loop in the diagram.
4. $\langle U \rangle = 1$ and $\langle \emptyset \rangle = 1$

Kauffman bracket example

Example)

1. Unlink

Kauffman bracket example

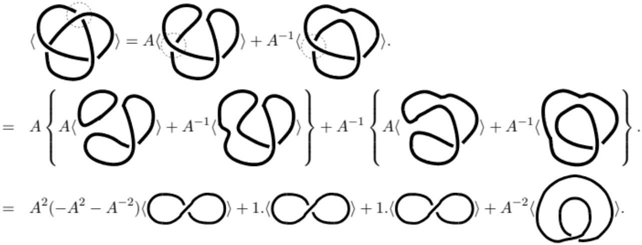
Example)

1. Unlink
2. Left Trefoil

Kauffman bracket example

Example)

1. Unlink
2. Left Trefoil



$$= (-A^4 + 1)(-A^3) + A^{-2}(-A^{-3}) = A^7 - A^3 - A^{-5}.$$

A state-sum model

The *Kauffman* bracket can be expressed by the explicit *state-sum* formula

$$\langle D \rangle = \sum_s \langle D|_s \rangle$$

where s runs over all states of D, and

$$\langle D|_s \rangle = A^{\sum_s} \left(-A^2 - A^{-2} \right)^{|s(D)|-1}$$

State

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A state s of a diagram D is an assignment of either $+1$ or -1 to each crossing. Form a new diagram sD by *resolving* or *splitting* the crossings of D.

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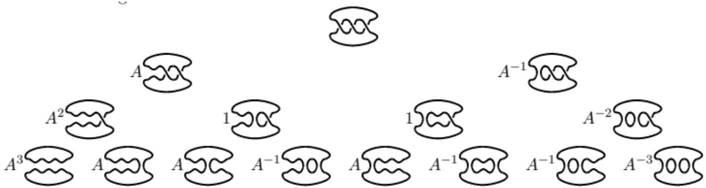
$\sum s$

For a state s, let $\sum s$ denote the sume of its values.

Appling the formula

Right Trefoil



$$A^3\left(-A^2-A^{-2}\right)+A+A+A^{-1}\left(-A^2-A^2\right)+A+$$
$$A^{-1}\left(-A^2-A^{-2}\right)+A^{-1}\left(-A^2-A^{-2}\right)+A^{-3}\left(-A^2-A^{-2}\right)^2=$$
$$-A^5+A^{-3}+A^{-7}$$

Appling the formula

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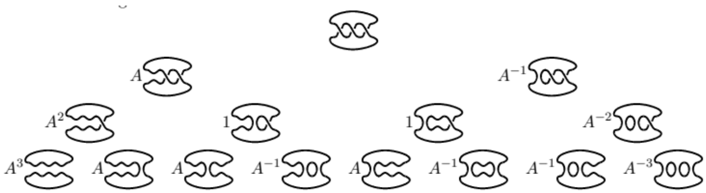

$A^3\left(-A^2-A^{-2}\right)+A+A+A^{-1}\left(-A^2-A^2\right)+A+$
$A^{-1}\left(-A^2-A^{-2}\right)+A^{-1}\left(-A^2-A^{-2}\right)+A^{-3}\left(-A^2-A^{-2}\right)^2=$
$-A^5+A^{-3}+A^{-7}$ We can see the mirror images through the Kauffman bracket.

Skein Relation

Is it invariant under the Reidemeister moves?

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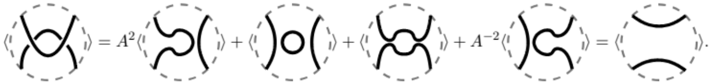

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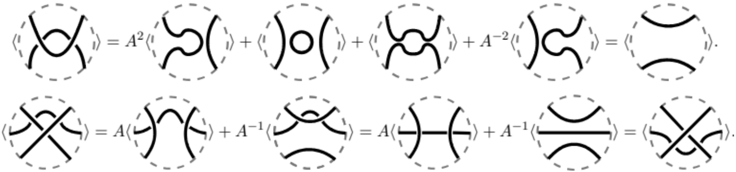

Skein Relation

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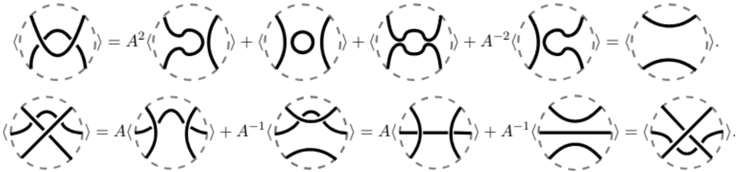
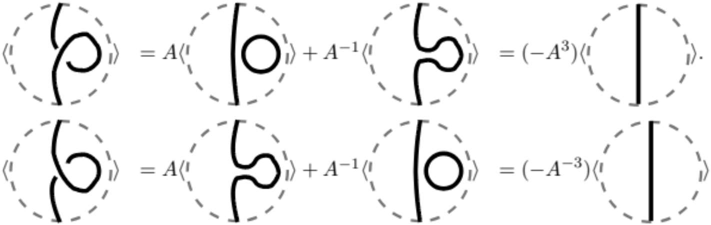
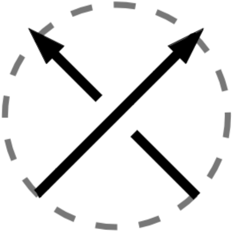


The Kauffman bracket is invariant under $R2$ and $R3$.

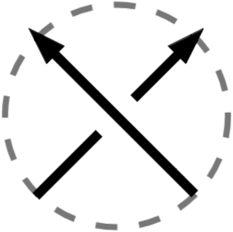
Reidermeister move 1



Signs of crossing



$+1$

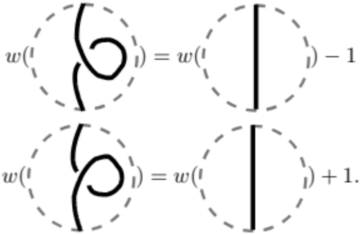


-1

Writhe

Definition

If D is an *oriented link diagram*, then the writhe $w(D)$ is just the sum of *the signs of all crossings* of D.



Writhe and Reidemeister move

The *writhe* of an oriented link diagram is invariant under $R2, R3$ but changes by ± 1 under $R1$.

Writhe and Reidemeister move

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Invariant of oriented links

The polynomial $f_D(A) = \left(-A^3\right)^{-w(D)} \langle D \rangle$ is invariant under all three Reidemester moves.

Proof.
Certainly it is invariant under R2, R3 since both the writhe and bracket are. All that remains is R1. If a diagram D is altered by the addition of a positive kink somewhere, then its Kauffman bracket multiplies by $\left(-A^3\right)$ and its writhe increases by 1; therefore $f_D(A)$ is unchanged. Similarly for the negative kink case.

Jones Polynomial

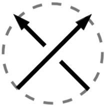
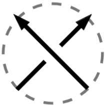
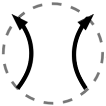
The *Jones polynomial* $V_L(t)$ of an *oriented link* L is the polynomial obtained by computing $f_D(A) = (-A^3)^{-w(D)} < D >$ for any diagram D of L, and the substituting $A = t^{-1/4}$.

Skein Relation

The *Jones Polynomial* satisfies

1. It is an invariant of oriented links lying in $\mathbb{Z}[t^{\pm 1/2}]$.
2. The *Jones polynomial* of the unknot is 1.
3. There is a skein relation

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0)$$

L_+ :  L_- :  L_0 : .

Example

1. $V(U_2)$

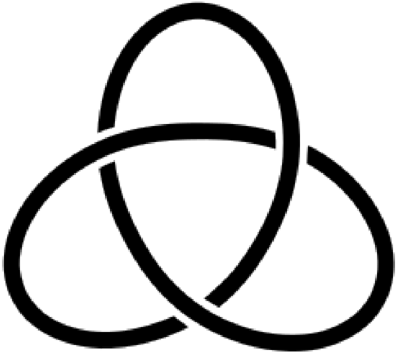
Example

1. $V(U_2)$
2. $V(H_+)$

Example

1. $V(U_2)$
2. $V(H_+)$
3. right-handed trefoil

Right-handed Trefoil



Remark

Left-handed trefoil $V(T_L) = -t^{-4} + t^{-3} + t^{-1}$

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Left-handed trefoil $V(T_L) = -t^{-4} + t^{-3} + t^{-1}$

Distinct from its mirror-image

The *Jones polynomial* of the mirror-image $\bar{L}$ of an oriented link L is the conjugate under $t \leftrightarrow t^{-1}$ of the polynomial of L.

$$V_{\bar{L}}(t) = V_L(t^{-1})$$

Characterisation

Suppose I is a $\mathbb{Z}[t^{\pm 1/2}]$-valued function of oriented links which satisfies

1. Isotopy invariance - it is an invariant of oriented links.
2. The Jones skein relation
   $t^{-1}I(L_+) - tI(L_-) = (t^{1/2} - t^{-1/2})I(L_0)$.
3. The normalisation $I(U) = 1$.

Then $I(L) = V(L)$ for all oriented links L.

Tait Conjecture

Tait conjecture

Any reduced diagram of an alternating link has the fewest possible crossings.

Conjectures

Unknotting problem

The unknot is the unique knot K with $V(K) = 1$.

Conjectures

Unknotting problem

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Volume Conjecture

https://en.wikipedia.org/wiki/Volume_conjecture

Other Invariants

Khobanov Homology

Categorification of the Jones polynomial

Others

- Alexander Polynomial
- Knot Floer Homology
- etc

References I

[1] URL https://knotplot.com/zoo/.

[2] URL https://mathweb.ucsd.edu/~justin/Roberts-Knotes-Jan2015.pdf.