Higher homotopy group & Poincaré Conjecture

0. Prerequisites

The prerequisites for this seminar are the fundamental group, covering spaces, singular homology, cohomology, CW complex, Categorical Algebra.

1. Higher homotopy groups

(X, x_0) : pointed topological space

Definition 1.1

$\pi_k(X, x_0)$ is a set of homotopy class of maps $f : (I^k, \partial I^k) \longrightarrow (X, x_0)$

equivalently, $f : (S^k, p) \longrightarrow (X, x_0)$

With a group operation $* : \pi_k \times \pi_k \longrightarrow \pi_k$: $[f] * [g] := [f * g]$

$$f * g := \begin{cases} f(2t_1, \cdots) & t_1 \in [0, \frac{1}{2}] \\ g(2t_1 - 1, \cdots) & t_1 \in [\frac{1}{2}, 1] \end{cases}$$

$\pi_k(X, x_0)$ is a group.

We already know that $\pi_1(X, x_0)$ can be nonabelian; figure-8.

However, $\pi_k(X, x_0)$ $(k \geq 2)$ are always abelian.

Theorem 1.2 [Eckmann–Hilton]

$\times, \otimes$: unital binary operations on X.

Suppose $(a \times b) \otimes (c \times d) = (a \otimes c) \times (b \otimes d)$, $\forall a, b, c, d \in X$.

Then, $\times, \otimes$ are exactly same operation and commutative & associative.

Proof) Left to a reader except units.

$1_\times, 1_\otimes$: units

$$1_\times = 1_\times \times 1_\times = (1_\times \otimes 1_\otimes) \times (1_\otimes \otimes 1_\times) = (1_\times \times 1_\otimes) \otimes (1_\otimes \times 1_\times)$$

$$= 1_\otimes \otimes 1_\otimes = 1_\otimes$$

Definition 1.3

$\otimes : \pi_k \times \pi_k \longrightarrow \pi_k : [f] \otimes [g] := [f \otimes g]$

$$f \otimes g := \begin{cases} f(t_1, 2t_2, \cdots) & t_2 \in [0, \frac{1}{2}] \\ g(t_1, 2t_2 - 1, \cdots) & t_2 \in [\frac{1}{2}, 1] \end{cases}$$

Theorem 1.4

$(\pi_k, *)$ is an abelian group.

Proof) $\forall f, g, h, k : (S^k, p) \to (X, x_0)$, $(f \otimes g) * (h \otimes k) = (f * h) \otimes (g * k)$

$\therefore ([f] \otimes [g]) * ([h] \otimes [k]) = ([f] * [h]) \otimes ([g] * [k])$

By Eckmann-Hilton argument, $*$ is abelian, as desired.

Theorem 1.1 [Lifting Criterion]

If $p : \tilde{X} \to X$ is a covering map, $f : (Y, y_0) \to (X, x_0)$, $p(\tilde{x}_0) = x_0$, then f lifts to a $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ with $p \circ \tilde{f} = f$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & (X, x_0) \end{tikzcd} \iff f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

Theorem 1.6

If $k \geq 2$, $\pi_k(S^1, p) = 0$

Proof) Recall $p : \mathbb{R} \to S^1$: covering map. $\forall f : S^k \to S^1$ lifts to a map $\tilde{f} : S^k \to \mathbb{R}$, by the Lifting Criterion.

Since $\mathbb{R}$ is contractible, $\tilde{f}$ is nullhomotopic, i.e., f is nullhomotopic.

2. Hurewicz isomorphism theorem.

Definition 2.1 [Hurewicz homomorphism]

$$\Phi : \pi_k(X, x_0) \longrightarrow H_k(X)$$
$$[f] \longmapsto f_*[S^k] \in H_k(X)$$

Since homology is homotopy invariance, Φ is well-defined.

Please check that Φ is a homomorphism.

Lemma 2.2

(X, x_0) : pointed space

i) $\forall k \geq 1$, if $f : (S^k, p) \longrightarrow (X, x_0)$ is nullhomotopic without fixing the base points, then $[f] = 0 \in \pi_k(X, x_0)$

ii) $\forall k \geq 2$, $f : \partial\Delta_{k+1} \longrightarrow X$: map sending $\forall (k-1)$-dimensional faces to x_0.

Then, $[f] = \displaystyle\sum^{*}_{\sigma \in F_k \Delta_{k+1}} [f|_\sigma] \in \pi_k(X, x_0)$

Sketch of Proof) i) Given a homotopy $F : S^k \times I \longrightarrow X$

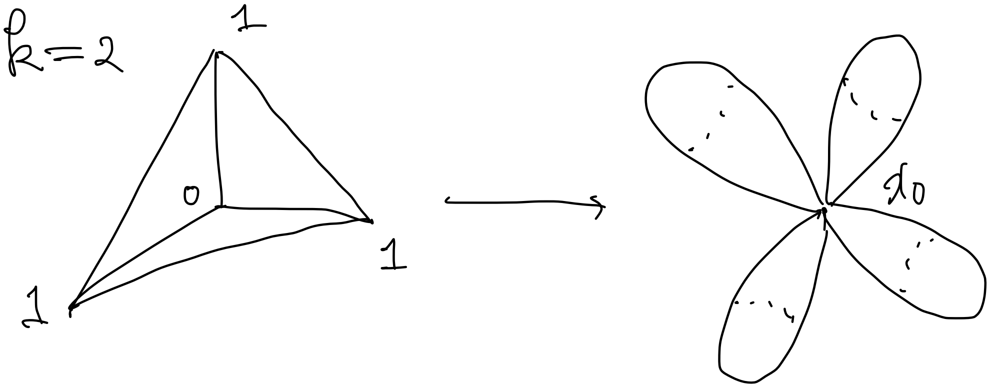
$F(s, 0) = f$

$F(s, 1) = \text{Const.}$

We can use $\gamma : I \to X$: trajectory of p to modify F to homotopy
$t \mapsto F(p, t)$
sending p to x_0 at all times.

ii) [Draw Picture]

$k=2$



Theorem 2.3 [Hurewicz isomorphism theorem]

$k \geq 2$. X : path-connected & $\pi_i(X, x_0) = 0$, $\forall i < k$.

Then Φ is an isomorphism, i.e., $\pi_k(X, x_0) \simeq H_k(X)$

Proof) Using Singular homology, define $\Psi : C_k(X) \to \pi_k(X, x_0)$

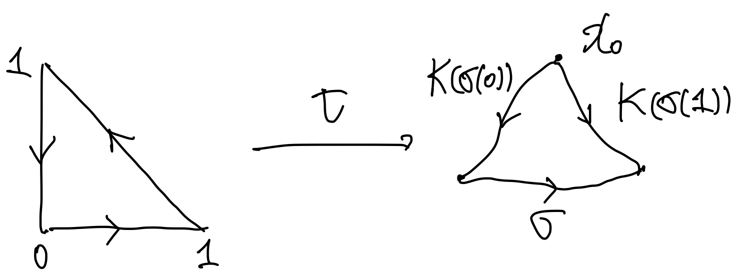
The idea is modify image of boundary via chain homotopy.

Define $K : C_i(X) \to C_{i+1}(X)$, $\forall 0 \leq i < k$ as follows.

Since X : path-connected, $\forall p \in X$, choose a path $K(p)$

from x_0 to p ; $K(p) : \Delta_1 \to X$ $\quad K(p)(0) = x_0$, $K(p)(1) = p$

For each 1-simplex $\sigma : \Delta_1 \to X$, $\exists \tau : \partial\Delta_2 \to X$ sending the three faces to σ, $K(\sigma(0))$, $K(\sigma(1))$.



Since X is simply connected, τ can be extended to $K(\sigma): \Delta_2 \longrightarrow X$ such that $\partial K(\sigma) + K(\partial\sigma) = \sigma$

Continuing by induction on i, if $1 \leq i < k$, then $\forall$ i-simplex $\sigma: \Delta_i \to X$, choose $(i+1)$-simplex $K(\sigma): \Delta_{i+1} \to X$ which sends the faces to σ, and the faces of $K(\partial\sigma)$ and therefore satisfies

$$\partial K(\sigma) + K(\partial\sigma) = \sigma$$

Finally if $\sigma: \Delta_k \to X$ is a k-simplex, then $\exists\ \tau: \partial\Delta_{k+1} \longrightarrow X$ sending the faces to σ, and the faces of $K(\partial\sigma)$.

Let q denote the face sent to x_0.

Identifying $(\partial\Delta_{k+1}, q) \simeq (S^k, p)$ this gives $\Psi(\sigma): (S^k, x_0) \longrightarrow (X, x_0)$
$\in \pi_k(X, x_0)$

Moreover, the sum of the faces is homologous to $\Psi(\sigma)$ regarded as a simplex, i.e., $\exists$ a simplex $K(\sigma)$ with

$$\partial K(\sigma) + K(\partial\sigma) = \sigma - \Phi(\Psi(\sigma))$$

While Ψ may depend on the above choices, we claim that Ψ induces a map on homology which is inverse to Φ.

Theorem 2.4

$$\pi_k(S^n) = \begin{cases} 0 & \text{if } k<n \\ \mathbb{Z} & \text{if } k=n \end{cases}$$

Proof) By the Hurewicz Isomorphism theorem, it's clear.

3. Dependence of π_k on the base point

We will show that if X is path connected, then the homotopy groups of X for different choices of base point are isomorphic.

If $\gamma : I \to X$: path from x_0 to x_1, define a map

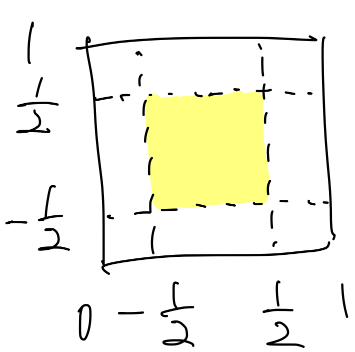
$$\Phi_\gamma : \pi_k(X, x_1) \longrightarrow \pi_k(X, x_0) \quad \text{as follows}$$

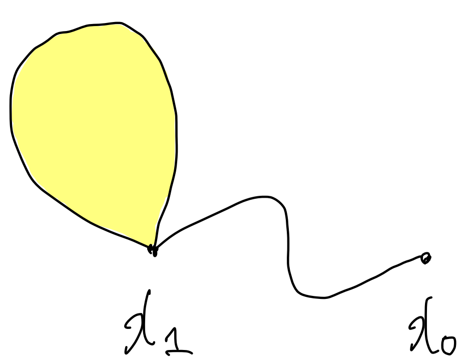
Let us temporarily reparametrize $I^k = [-1,1]^k$

If $f : [-1,1]^k \to X$ & $t = (t_1, \cdots, t_k) \in I^k$

let $m = \max\{|t_i|\}$ and define

$$\Phi_\gamma([f(t)]) := \left[\begin{cases} f(2t) & m \leq \frac{1}{2} \\ \gamma(2(1-m)) & m \geq \frac{1}{2} \end{cases} \right]$$





Please check that Φ_γ is group homomorphism.

The following facts are easy to see

i) If $\gamma \simeq_h \gamma'$, then $\Phi_\gamma = \Phi_{\gamma'} : \pi_k(X, x_0) \to \pi_k(X, x_1)$

ii) If γ : path from x_0 to x_1, γ' : path from x_1 to x_2

, then $\Phi_{\gamma_2 * \gamma_1} = \Phi_{\gamma_2} \circ \Phi_{\gamma_1} : \pi_k(X, x_2) \longrightarrow \pi_k(X, x_0)$

iii) If γ : constant path at x_0, then $\Phi_\gamma = i_{\pi_k(X, x_0)}$

Theorem 3.1

If X : path connected, then $\pi_k(X, x_0) \simeq \pi_k(X, x_1)$

4. Poincaré conjecture

Theorem 4.1. [Poincaré duality]

If M is n-dimensional orientable closed manifold, then

$$H_{n-k}(M) \simeq H^k(M)$$

Definition 4.2 [Hom, Ext Functor]

A, B : R-module

$Hom_R(A, B) :=$ {Homomorphism from A to B}

For a fixed A, Hom_R is left exat functor.

For a injective resolution, $0 \to B \to I^0 \to I^1 \to \cdots$

Form the cochain complex, $0 \to Hom_R(A, I^0) \to Hom_R(A, I^1) \to$

$Ext_R^i(A, B) :=$ ith cohomology of this complex

Ext_R^i is right derived functor from Hom_R

Theorem 4.3 [Universal coefficient theorem for cohomology]

R: PID, G: R-module, $\exists$ short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{i-1}(X;R), G) \to H^i(X;G) \to \mathrm{Hom}_R(H_i(X;R), G) \to 0$$

and the sequence splits.

Theorem 4.4 [Homology Whitehead theorem]

X, Y : simply connected CW complexes

$f: X \to Y$.

If $f_*: H_n(X) \to H_n(Y)$ is isomorphism $\forall n$,

then f: homotopy equivalence.

Theorem 4.5

Every simply connected, orientable, closed 3-manifold is homotopy equivalent to S^3.

Proof) X: simply connected, orientable, closed 3-manifold

Then, $H_0(X) \simeq \mathbb{Z}$, $H_1(X) = \pi_1(X) = 0$

Since X is orientable, $H_3(X) \simeq \mathbb{Z}$.

By Poincaré duality, $H_2(X) \simeq H^1(X)$

By Universal coefficient theorem,

$$H^1(X) \simeq \mathrm{Hom}(H_1(X); \mathbb{Z}) \oplus \mathrm{Ext}(H_0(X); \mathbb{Z}) = 0$$

So $H_2(X) = 0$.

By Hurewicz theorem, $\Phi : \pi_3(X) \longrightarrow H_3(X)$

is an isomorphism

Let $f : S^3 \to X$ represent a generator of $\pi_3(X)$

Then, $f_*: H_n(S^3) \longrightarrow H_n(X)$ are isomorphisms

It follows that f is a homotopy equivalence

Theorem 4.6 [Poincaré Theorem]

Such X is, in fact, homeomorphic to S^3.