Nets and Riemann Integration

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Motivation

Riemann Integral $=$ Limit of a Riemann sum(?)

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Directed Sets

Definition

A **directed set** is a set A equipped with a binary relation $\preceq$ such that

- $a \preceq a$ for all $a \in A$.
- If $a \preceq b$ and $b \preceq c$ then $a \preceq c$.
- For any $a, b \in A$ there exists $c \in A$ such that $a \preceq c$ and $b \preceq c$.

Examples

- The set of positive integers $\mathbb{N}$, with $j \preceq k$ iff $j \leq k$.
- The set $\mathbb{R}^n \setminus \{a\}$ $(a \in \mathbb{R}^n)$, with $x \preceq y$ iff $\|x - a\| \geq \|y - a\|$.
- The set $\mathcal{N}$ of all neighborhoods of a point x in a topological space X, with $U \preceq V$ iff $U \supseteq V$.

Directed Sets

Definition

A **directed set** is a set A equipped with a binary relation $\preceq$ such that

- $a \preceq a$ for all $a \in A$.
- If $a \preceq b$ and $b \preceq c$ then $a \preceq c$.
- For any $a, b \in A$ there exists $c \in A$ such that $a \preceq c$ and $b \preceq c$.

Examples (Continued)

$\mathcal{P}[a,b]$: The set of all partitions of $[a,b]$. $(a, b \in \mathbb{R}, a < b)$
$(i.e.\ \ \mathcal{P}[a,b]$ is the set of all finite subsets of $[a,b]$ containing $a, b.)$
For $P_1, P_2 \in \mathcal{P}[a,b]$, define

- $P_1 \preceq_1 P_2$ iff $P_1 \subseteq P_2$.
- $P_1 \preceq_2 P_2$ iff $\|P_1\| \geq \|P_2\|$.

Nets

Definition

A **net** in a set X is a mapping $\alpha \mapsto x_\alpha$ from a directed set A into X.
We denote it by $\langle x_\alpha \rangle_{\alpha \in A}$.

Definition

Let X be a topological space and $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X.
We say that $\langle x_\alpha \rangle_{\alpha \in A}$ **converges** to $x \in X$, if for each neighborhood U of x, there exists $\alpha_0 \in A$ such that $x_\alpha \in U$ for all $\alpha \succeq \alpha_0$.

Examples

- A sequence in a topological space
- A function from $\mathbb{R}$ to $\mathbb{R}$

Darboux Integral

Fix a bounded $f : [a, b] \to \mathbb{R}$, and let

$$P = \{x_0(= a), x_1, ..., x_{n-1}, x_n(= b)\}$$

be a partiton of $[a, b]$. For each $i = 1, ..., n$, put

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}, \qquad m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\},$$

and define

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}), \qquad L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Let

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}[a, b]\}, \qquad L(f) = \sup\{L(f, P) : P \in \mathcal{P}[a, b]\}.$$

Definition

If $U(f) = L(f)$, we say that f is **Darboux integrable** on $[a, b]$, and denote its common value by $\int_a^b f(x)dx$.

Darboux Integral

Recall that the set $\mathcal{P}[a,b]$, together with a binary relation $\preceq_1$ defined by

$$P_1 \preceq_1 P_2 \text{ iff } P_1 \subseteq P_2 \qquad (P_1, P_2 \in \mathcal{P}[a,b])$$

is a directed set.

Proposition

For a bounded $f : [a,b] \to \mathbb{R}$ and $A \in \mathbb{R}$, the following are equivalent.

1. f is Darboux integrable and $\int_a^b f(x)dx = A$.
2. Two nets

$$P \mapsto U(f,P), P \mapsto L(f,P) : \mathcal{P}[a,b] \to \mathbb{R}$$

converge to the same real number A.

Riemann Integral

Definition

1. A **tagged partition** of $[a, b]$ is a partition

$$\dot{P} = \{x_0, x_1, ..., x_{n-1}, x_n\}$$

together with a choice of sample points in each sub-interval; numbers $(t_i)_1^n$ with $t_i \in [x_{i-1}, x_i]$.

2. If $f : [a, b] \to \mathbb{R}$ is bounded and $\dot{P}$ is a tagged partition of $[a, b]$, we define the **Riemann sum** of f as

$$\mathcal{R}(f, \dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

3. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded.
We say that f is **Riemann integrable** on $[a, b]$, if there exists $A \in \mathbb{R}$ with the following property:
For a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\mathcal{R}(f, \dot{P}) - A| < \epsilon$$

for all tagged partition $\dot{P}$ of $[a, b]$ with $\|\dot{P}\| < \delta$.
In this case, the number A is called the **Riemann integral** of f, and is denoted by $\int_a^b f(x)dx$.

Riemann Integral

Remark : Reformulation of Riemann's definition

f is **Riemann integrable** on $[a, b]$ if and only if there exists $A \in \mathbb{R}$ with the following property:
For a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|U(f, P) - A| < \epsilon, \qquad |L(f, P) - A| < \epsilon$$

for all partition P of $[a, b]$ with $\|P\| < \delta$.

Riemann Integral

Recall that the set $\mathcal{P}[a,b]$, together with a binary relation $\preceq_2$ defined by

$$P_1 \preceq_2 P_2 \text{ iff } \|P_1\| \geq \|P_2\| \qquad (P_1, P_2 \in \mathcal{P}[a,b])$$

is a directed set.

Proposition

For a bounded $f : [a,b] \to \mathbb{R}$ and $A \in \mathbb{R}$, the following are equivalent.

1. f is Riemann integrable and $\int_a^b f(x)dx = A$.
2. Two nets
$$P \mapsto U(f,P), P \mapsto L(f,P) : \mathcal{P}[a,b] \to \mathbb{R}$$
converge to the same real number A.

Equivalence of two definitions

Remark : Comparison between Darboux and Riemann

Two nets

$$P \mapsto U(f,P), P \mapsto L(f,P) : \mathcal{P}[a,b] \to \mathbb{R}$$

must converge to the same real number A, w.r.t. $\preceq$, where

1. (Darboux) $P_1 \preceq P_2$ iff $P_1 \subseteq P_2$.
2. (Riemann) $P_1 \preceq P_2$ iff $\|P_1\| \geq \|P_2\|$.