Geometric Flow and Curve Shortening Flow

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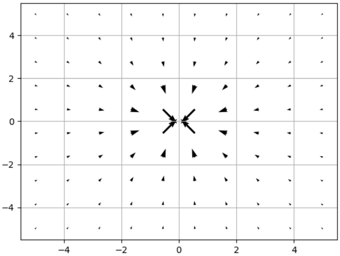
Flow

$F : \mathbb{R}^2 \to \mathbb{R}^2$.

For example, Gravitational Force Field.

$$G(x, y) = \left(-x/\sqrt{x^2 + y^2}, -y/\sqrt{x^2 + y^2}\right)$$



$$\dot{x}(t) = G(x(t))$$

Gradient

Let $F : \mathbb{R}^2 \to \mathbb{R}$.
Gradient of $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right)$.

What is the geometric meaning of gradient?

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Let's try it!

Gradient Flow

We can use this idea to minimizing or maximizing something. For example, Artificial Neural Network in Deep Learning, we use gradient to optimize loss function.

Gradient Descent

$$x_{t+1} = x_t - \eta \nabla F\left(x_t\right)$$

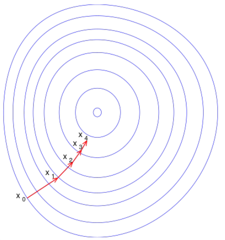

Figure 1: Gradient Descent

Gradient Descent

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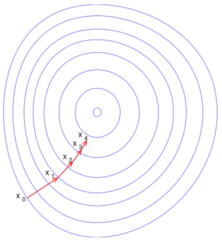

Figure 1: Gradient Descent

Also there are many different methods, such as stochastic gradient descent, AdaGrad, Adam, . . .

Like these cases, we can think *flow* as changing or moving from the states.

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Curve

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Simple Curve

Curve does not cross itself.

Arc Length

Arc length $S(p)$

Arc length $S(p) = \int_0^p \left\| \frac{d}{du}\gamma(u) \right\| du$

Complete Curve

$\lim_{\|p\| \to \infty} S(p) = \infty$

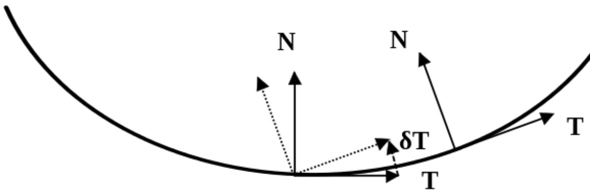

Figure 2: T, N and Curvature

Curvature of the curve

$$\frac{ds}{dp} = \left\|\gamma'(p)\right\| > 0.$$

$$T := \gamma_s = \frac{d}{ds}\gamma = \frac{dp}{ds}\frac{d}{dp}\gamma = \frac{\gamma'(p)}{\left\|\gamma'(p)\right\|}$$

Curvature of the curve

$$\frac{ds}{dp} = \left\| \gamma' \left(p \right) \right\| > 0.$$

$$T := \gamma_s = \frac{d}{ds}\gamma = \frac{dp}{ds}\frac{d}{dp}\gamma = \frac{\gamma' \left(p \right)}{\left\| \gamma' \left(p \right) \right\|}$$

After that choose normal vector N and $\gamma_{ss} = \frac{\langle \gamma'', N \rangle N}{\left\| \gamma' \right\|^2} = \kappa N$

Convex Set

Convex Set

Let S be a vector space or an affine space over the real numbers, or, more generally, over some ordered field (this includes Euclidean spaces, which are affine spaces). A $C \subset S$ is convex if, for all $x, y \in C$, the line segment connecting x and y is included in C.

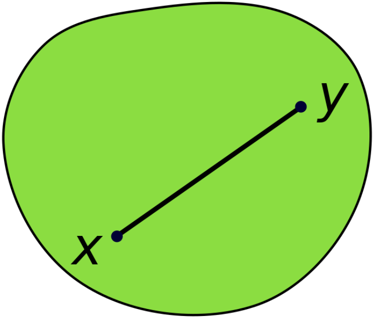

Figure 3: Convex Set

Convex Body

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A convex body in $\mathbb{R}^n$ is a compact convex set with non-empty interior.

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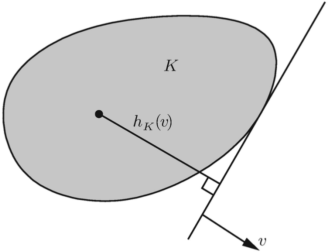

Figure 4: Convex Body

For someone(CW) who wants...

Here is the example of Convex Set but not Convex Body.

Let M be the set of all points in l_2, satisfying $\sum_{n=0}^{\infty} n^2 x_n^2 \leq 1$

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Geometric Flow

We can also consider the "Geometric Flow" similarly before.

$$\frac{\partial}{\partial t}\gamma$$

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Let's see through the simulation!

Curve Shortening Flow

Definition

Let $\{\Gamma_t\}_{t\in[0,T]}$ be a one-parameter family of complete curve with C^∞ immersion, $\gamma : \mathbb{R} \times [0, T] \to \mathbb{R}^2$ satisfying $\gamma_t = \gamma_{ss} = \kappa N$. Then $G = \bigcup_{t\in[0,T]} \Gamma_t \times \{t\} \subseteq \mathbb{R}^2 \times [0, T]$ is a Curve Shortening Flow.

Example

Let $\Gamma_0 = \{\|x\| = r\} \Rightarrow \Gamma_t = \left\{\|x\|^2 = r^2 - 2t\right\}$ is a CSF. Why?

Curve Shortening Flow

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Example

Let $\Gamma_0 = \{\|x\| = r\} \Rightarrow \Gamma_t = \left\{\|x\|^2 = r^2 - 2t\right\}$ is a CSF. Why?

$$\gamma(\theta, t) = \sqrt{r^2 - 2t}\,(\cos\theta, \sin\theta)$$

$$\Rightarrow \gamma_t = \gamma_{\theta\theta} = -\frac{(\cos\theta, \sin\theta)}{\sqrt{r^2 - 2t}}$$

Convex Complete Curve

Convex Complete Curve

If $\Gamma \subset \mathbb{R}^2$ is a convex complete curve if $\Gamma = \partial K$ for some convex body K.

Support function

Given a convex body $K \subset \mathbb{R}^2$, we can define support function

$$S(\theta) := \sup_{x \in K} \langle x, (\cos\theta, \sin\theta) \rangle$$

We can just consider $S(\theta) = -\langle \gamma(\theta), N(\theta) \rangle$.

Reparametrization

If ∂K is C^∞ and closed, ∂K can be reparametrizated by $\gamma : S^1 \to \mathbb{R}^2$ with $N(\theta) = (-\cos\theta, \sin\theta)$ such as

$$\gamma = (S\cos\theta - S_\theta \sin\theta, S\sin\theta + S_\theta\cos\theta).$$

Then we can express the curvature $\kappa = \frac{d\theta}{ds}$.
Also, using chain rule, we can derive $\kappa = \frac{1}{S_{\theta\theta}+S} > 0$

Convex closed CSF

Suppose Γ_t is a convex closed CSF. Then $\gamma_t = \kappa N$ where

$$\kappa(\theta, t) = \left\langle \frac{\partial}{\partial t}\gamma(\theta, t), N(\theta) \right\rangle = -\frac{\partial}{\partial t} S.$$

$$S_t = -\kappa = -\frac{1}{S_{\theta\theta} + S}$$

$$\kappa_t = \kappa^2 \kappa_{\theta\theta} + \kappa^3.$$

So we also can analyze through the semi-linear heat equation!

Properties

There are some interesting properties of the convex closed CSF.

Area

Area of the region enclosed by Γ, $A(\Gamma_t)$ decreases in rate of 2π i.e. $\frac{d}{dt}A(\Gamma_t) = -2\pi$.

So $A(\Gamma_t) = A(\Gamma_0) - 2\pi t$.
And let $\bar{T} := A(\Gamma_0)/2\pi$

Local Existence

Convex Body

During $t < \bar{T}$, convex body $K_{t_2} \subset K_{t_1}$ where $t_1 \leq t_2$ and $\Gamma_t = \partial K_t$.

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Unit Circle[2]

Also if we resclaed the area as 1, there is a $t_0 < \bar{T}$ such that Γ_{t_0} is a unit circle.

$t \to \bar{T}$

When $t \to \bar{T}$, $\Gamma_t \to$ a point!

Application in mathematics

We can use the CSF to isoperimetric inequality[3].

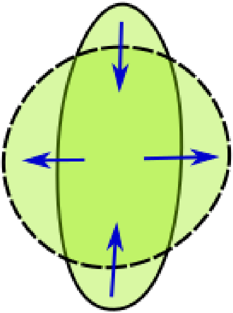

Figure 5: Isoperimetric Inequality

Applications

- Annealing metal sheets
- Reaction–diffusion

More

Related Geometric Flows

- Mean Curvature Flow - Area Minimizing
- Ricci Flow - as the solution of the Poincare's conjecture.
- Inverse Mean Curvature Flow

Numerical Approximation

Also it can be numerically approximated by Finite Difference Method well.

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Thank you