Converse of Intermediate Value Theorem : From Falsity to Conway 13 Function

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Definition 0.1

Let $a \in \mathbb{R}$. For each $r > 0$, the <u>open ball</u> centered at a of radius r is the set of points

$$B_r(a) := \{x \in \mathbb{R} : |x - a| < r\}.$$

Definition 0.2

Let $f : E \to \mathbb{R}$ be a real function where E is a nonempty subset of $\mathbb{R}$. f is said to be <u>continuous</u> at a point $a \in E$ if and only if for $x \in E$,

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } |x - a| < \delta \; \to |f(x) - f(a)| < \epsilon.$$

In other words,

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } f[B_\delta(a)] \subseteq B_\epsilon(f(a)).$$

1. Introduction

Intermediate Value Theorem (as known as IVT) is a well-known theorem for students who have studied calculus or analysis.

Theorem 1.1 (Intermediate Value Theorem)

Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if y_0 is a number such that $f(a) < y_0 < f(b)$, then there exists a point $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

The Intermediate Value Theorem can be applicated to some useful ways. For example, we can show existence of a solution of some equation by using Intermediate Value Theorem when the equation is not soluble.

Example 1.2

Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be a continuous real function defined by

$$f(x) = 4\sin x - 1.$$

Then there exists some $p \in (0, \frac{\pi}{2})$ satisfying $f(x) = 0$.

Proof. We can easily check that $f(0) = -1$, $f(\frac{\pi}{2}) = 3$ and

$$f(0) < 0 < f\left(\frac{\pi}{2}\right).$$

Since f is continuous on $[0, \frac{\pi}{2}]$, there exists a point $p \in (0, \frac{\pi}{2})$ such that $f(p) = 0$.

2. Converse of IVT

Conjecture 2.1

Let I be a nonempty open interval of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a real function. If for any $x_1,\ x_2 \in I$ satisfying $x_1 < x_2$, and for any $y \in \mathbb{R}$ between $f(x_1)$ and $f(x_2)$ there exists some $x \in (x_1, x_2)$ such that $f(x) = y$, then f is continuous on I.

Lemma 2.2

Let E be a nonempty subset of $\mathbb{R}$, $a \in E$ and let $f : E \to \mathbb{R}$ be a continuous function at $x = a$. If x_n converges to a and $x_n \in E$, then

$$f(x_n) \to f(a) \quad \text{as} \quad n \to \infty.$$

Example 2.3

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function defined by

$$f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(i) f satisfies the conclusion of Theorem 1.1.
(ii) f is discontinuous at $x = 0$.

Proof. (i) Let $a, b \in \mathbb{R}$ and $a < b$. Since f is continuous on $(-\infty, 0)$ and $(0, \infty)$, f satisfies the conclusion of Theorem 1.1 on any closed interval $[a, b]$ when $[a, b] \subseteq (-\infty, 0)$ or $[a, b] \subseteq (0, \infty)$.

(Continued :)
Now suppose that $a < 0$ and $b > 0$. (i.e. $[a, b] \not\subseteq (-\infty, 0)$ and $[a, b] \not\subseteq (0, \infty)$.) Since $f[\mathbb{R}] = [-1, 1]$, the proof is complete if we find $c, d \in \mathbb{R}$ $(c < d)$ such that $[c, d] \subseteq [a, b]$ and $f[c, d] = [-1, 1]$.
Let $x_n = \frac{2}{(2n+1)\pi}$. Then x_n is monotone decreasing to 0 and there exists some p such that $x_{p+1} < a \leq x_p$. Let $c = x_{p+2}$ and $d = x_{p+1}$. Since

$$f(x_n) = \sin\left(n\pi + \frac{\pi}{2}\right) = (-1)^n,$$

$f[c, d] = [-1, 1]$ and f satisfies the conclusion of Theorem 1.1.

(ii) Suppose f is continuous at $x = 0$. Since $x_n \in \mathbb{R}$ and $x_n \to 0$ but $f(x_n)$ does not converge, this is a contradiction to Lemma 2.2. Thus f is discontinuous at $x = 0$.

f in Example 2.3 is a nice counterexample to Conjecture 2.1, considering that it is easy to understand why this example is a counterexample of the previous proposition.

However, this example is a bit weak considering that f can be a counterexample only when $x = 0$. In other words, f satisfies Conjecture 2.1 on $(-\infty, 0)$ and $(0, \infty)$. Hence, we may want to find a function that can be a counterexample of Conjecture 2.1 on any interval. We will begin with the following definition.

Definition 2.4

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and let I be any nonempty open interval of $\mathbb{R}$. f is said to be a <u>$(\star)$-function</u> if

$$f[I] = \mathbb{R} \quad \forall I \subseteq \mathbb{R}.$$

This definition leaves us two questions. First, we might wonder that if there exists any $(\star)$-function on $\mathbb{R}$. The existence of $(\star)$-function will be proved later in Theorem 3.7 and Theorem 4.6. We might also wonder what Definition 2.4 is for. The answer follows from the following theorem.

Theorem 2.5

Let $f : \mathbb{R} \to \mathbb{R}$ be a $(\star)$-function. Then
(i) f satisfies the conclusion of Theorem 1.1 on any interval $I \in \mathbb{R}$.
(ii) f is discontinuous at every real number x.

Proof. (i) Let $I = [a, b]$ and let $[c, d] \subseteq I$. Since f is a $(\star)$-function, $f[c, d] = \mathbb{R}$ and there exists some $p \in [c, d]$ such that $f(p)$ lies between $f(a)$ and $f(b)$ and the proof is complete.

(ii) Suppose that there exists some $\alpha \in \mathbb{R}$ such that $f(x)$ is continuous at $x = \alpha$. Let $B_\delta(\alpha)$ be a ball of radius δ centered at α and let $B_\epsilon(f(\alpha))$ be an open ball of radius ϵ centered at $f(\alpha)$. Then for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$f[B_\delta(\alpha)] \subseteq B_\epsilon(f(\alpha)).$$

(Continued :)
Since f is a $(\star)$-function, $f[N_\delta(\alpha)] = \mathbb{R}$ and then for all $\epsilon > 0$,

$$\mathbb{R} \subseteq B_\epsilon(f(\alpha)).$$

This is a contradiction. Thus f is discontinuous at every real number x.

Theorem 2.5 shows that if f is a $(\star)$-function, then f is a counterexample of Conjecture 2.1 on any interval. Hence we may need to show that there exists some $(\star)$-function f on $\mathbb{R}$.

Lemma 3.1

Let $\langle \mathbb{R}/\mathbb{Q}, +\rangle$ be a factor group and let $a, b \in \mathbb{R}$. Define a relation $\sim$ by

$$a \sim b \quad \textit{iff} \quad a + \mathbb{Q} = b + \mathbb{Q} \quad \textit{in} \quad \mathbb{R}/\mathbb{Q}. \quad (\textit{i.e. } a - b \in \mathbb{Q}.)$$

Then $\sim$ is an equivalence relation.

Proof. Since $a - a = 0 \in \mathbb{Q}$, $a \sim a$ and $\sim$ is reflexive. Next, suppose that $a \sim b$. Then $a - b \in \mathbb{Q}$ and $b - a \in \mathbb{Q}$. Hence $b \sim a$ and $\sim$ is symmetric. Now we also suppose that $b \sim c$. Then $a - c = (a - b) + (b - c) \in \mathbb{Q}$ and we get $a \sim c$. Thus $\sim$ is transitive and the proof is complete.

Since $\sim$ we defined in Lemma 3.1 is an equivalence relation, we can form an equivalence classes of $\mathbb{R}$ given by $\sim$. We will define V_{set} as a set of such equivalence classes.

Definition 3.2

Let V_{set} be a set and let $\langle \mathbb{R}/\mathbb{Q}, + \rangle$ be a factor group.
(i) For $a \in \mathbb{R}$, $a \in V_{\text{set}}$ if and only if $a + \mathbb{Q} \in \mathbb{R}/\mathbb{Q}$.
(ii) If $a, b \in V_{\text{set}}$ and $a \sim b$, then $a = b$.
Such sets are called to be a Vitali set.

In other words, we pick one element from each equivalence classes of $\sim$ by axiom of choice and form a Vitali set. Since equivalence classes of $\mathbb{R}$ is a partition of $\mathbb{R}$, we can consider all elements of V_{set} as linearly independent vectors. This leads us to a following lemma.

Lemma 3.3

Let V_{set} be a Vitali set. Then V_{set} is a basis of $\mathbb{R}$ over $\mathbb{Q}$.

Proof. As we mentioned already, all elements of V_{set} are linearly independent. Let $V = \cup_{a \in V_{\text{set}}}(\mathbb{Q} + a)$. If V covers $\mathbb{R}$, then $V \supseteq \mathbb{R}$. Since $a \in \mathbb{R}$ for all $a \in V$, $V \subseteq \mathbb{R}$. Hence $V = \mathbb{R}$ and it is sufficient to show that V_{set} is a basis of $\mathbb{R}$. Suppose V is not a cover of $\mathbb{R}$. Then $V \subset \mathbb{R}$ and there exists some $r \in \mathbb{R}$ such that $r \notin V$. This is a contradiction because V is union of a partition of $\mathbb{R}$ and the proof is complete.

Lemma 3.4

Let V_{set} be a Vitali set. Then

$$|V_{set}| = |\mathbb{R}|.$$

Proof. Since V_{set} is a basis of $\mathbb{R}$ over $\mathbb{Q}$,

$$|\mathbb{Q}| \cdot |V_{\text{set}}| = |\mathbb{R}|.$$

Note that $\mathbb{Q}$, V_{set}, $\mathbb{R}$ are all infinite sets. By basic cardinal arithmetic, we have

$$|\mathbb{R}| = \max\{|\mathbb{Q}|, |V_{\text{set}}|\}.$$

Now it remains to show that $\max\{|\mathbb{Q}|, |V_{\text{set}}|\} = |V_{\text{set}}|$. Since $\mathbb{Q}$ is countable, it is sufficient to show that V_{set} is uncountable. Suppose V_{set} is countable. Then $\mathbb{Q} \times V_{\text{set}}$ is countable and this is a contradiction because $\mathbb{R}$ is uncountable. Hence V_{set} is uncountable and the proof is complete.

As we showed in Lemma 3.4, a Vitali set has the same cardinality with $\mathbb{R}$. Hence there exists a bijection between V_{set} and $\mathbb{R}$. Now we are almost ready to define a $(\star)$-function. Before definition, we will quickly go through a following lemma.

Lemma 3.5 (Density of Rationals)

Let $a, b \in \mathbb{R}$ with $a < b$. Then there exists some $q \in \mathbb{Q}$ such that $a < q < b$.

Lemma 3.6

For any open interval $I \subseteq \mathbb{R}$, there exists some Vitali set V_{set} such that

$$V_{set} \subseteq I.$$

Proof. Let $a, b \in \mathbb{R}$ $(a < b)$, $I = (a, b)$ and let V'_{set} be some Vitali set. Let $\sim$ be a relation defined as Lemma 3.1. By Lemma 3.5 and axiom of choice, we can form a Vitali set V_{set} such that for all $v' \in V'_{\mathrm{set}}$ there exists some $v \in V_{\mathrm{set}}$ satisfying $v \sim v'$ and $a < v < b$. Thus we have $V_{\mathrm{set}} \subseteq I$.

Theorem 3.7

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Then there exists some $(\star)$-function f.

Proof. Let I be any open interval of $\mathbb{R}$. By Lemma 3.6, there exists some Vitali set V_{set} such that $V_{\text{set}} \subseteq I$. Since there exists a bijection between V_{set} and $\mathbb{R}$ by Lemma 3.4, there exists some f such that

$$f[V_{\text{set}}] = \mathbb{R}.$$

Since I is any open interval of $\mathbb{R}$, f is a $(\star)$-function and the proof is complete.

3. Vitali Set

Vitali set allows us define a $(\star)$-function in natural way. However, this counterexample might also be a bit weak, considering it only shows existence of a counterexample. Hence we may wonder how the $(\star)$-function actually looks.

4. Conway's Base 13 Function

In this section, we are going to use numbers with base 13. Let the distinct base 13 symbols be

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C.$$

Consider A as 10 in base 10, B as 11 in base 10, C as 12 in base 10. (i.e. $A_{13} = 10_{10}$, $B_{13} = 11_{10}$, $C_{13} = 12_{10}$.) For example, any real number $r \in [0, 1)$ can be expressed as

$$r = 0.c_1 c_2 c_3 \cdots c_n \cdots_{13} \tag{1}$$

where $c_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C\}$ for all i. Now we start this section with couple definitions.

Definition 4.1

Let r be defined on $[0, 1)$ as (1).
(i) For all $N \in \mathbb{N}$, if there exists some $n > N$ such that $c_n \neq C$, r is said to be <u>normal</u>.
(ii) A <u>tail</u> of r is a sequence $c_n c_{n+1} c_{n+2} \cdots$ for some $n \geq 1$.

Definition 4.2

Let r be defined on $[0, 1)$ as (1). A tail $x_0 x_1 x_2 \cdots x_n y_0 y_1 y_2 \cdots$ of r is said to be <u>special</u> if
(i) $x_0 = A$ or $x_0 = B$.
(ii) $y_0 = C$.
(iii) For all $i > 0$, $x_i, y_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

By Definition 4.2, a special tail has only x_0, y_0 as A, B or C. This leads us to a following lemma.

Lemma 4.3

Let r be defined on $[0,1)$ as (1). Then r has at most one special tail.

Proof. Suppose r has more than one special tail. Let $x_0x_1x_2\cdots x_ny_0y_1y_2\cdots$ be a first special tail of r and let $z_0z_1z_2\cdots z_nw_0w_1w_2\cdots$ be a second special tail of r. Then $y_k = z_0$ for some k. This is a contradiction because $y_k \in \{0,1,2,3,4,5,6,7,8,9\}$ and $z_0 \in \{A, B\}$. Thus r has at most one special tail.

Some might think it is weird to make definitions and theorems in base 13. It is true that base 13 representations are not useful on usual circumstances. But it allows us to define a $(\star)$-function with output in base 10, which is familiar to us. This cannot be done without base 13 representation. Also note that Lemma 4.3 shows any $r \in [0, 1)$ has at most one special tail. Moreover, considering $x_0 \in \{A, B\}$ and $y_0 = C$, r has only three cases: r has no special tail, r has a special tail starting with A, or r has a special tail starting with B. (Here, we do not consider other x_i, y_is.)

Definition 4.4 (Conway's Base 13 Function)

Let $x = 0.c_1c_2c_3\cdots c_n\cdots_{13}$ be a base 13 normal number in $[0,1)$ and let $f_c : [0,1) \to \mathbb{R}$ be a real function defined by

$$f_c(x) := \begin{cases} 0 & \text{if (i)}, \\ x_1x_2x_3\cdots x_n.y_1y_2y_3\cdots & \text{if (ii)}, \\ -x_1x_2x_3\cdots x_n.y_1y_2y_3\cdots & \text{if (iii)} \end{cases}$$

where
(i) : x has no special tail,
(ii) : x has a special tail $x_0x_1x_2\cdots x_ny_0y_1y_2\cdots$ and $x_0 = A,\ y_0 = C$,
(iii) : x has a special tail $x_0x_1x_2\cdots x_ny_0y_1y_2\cdots$ and $x_0 = B,\ y_0 = C$.
Then f_c is well-defined by Lemma 4.3. Such f_c is called to be Conway's base 13 function.

Lemma 4.5

Let f_c be Conway's base 13 function and let I be any nonempty open interval of $[0,1)$. Then

$$f_c[I] = \mathbb{R} \quad \forall I \subseteq [0,1).$$

Proof. Let $a = 0.a_1a_2a_3\cdots_{13}$, $b = 0.b_1b_2b_3\cdots_{13}$ $(0 \leq a < b < 1)$ and $I = (a,b)$. Let
(i) i be the least number such that $a_i < b_i$.
(ii) j be the least number greater than i such that $a_j < C$.
Since $a < b$, such i exists and since we are working with normal numbers, such j exists. Now let

$$x = x_1x_2x_3\cdots x_n.y_1y_2y_3\cdots_{10}$$

and let

$$c = 0.a_1a_2\cdots a_iC_{i+1}C_{i+2}\cdots C_jAx_1x_2\cdots x_nCy_1y_2\cdots_{13}$$

where $C_{i+1} = C_{i+2} = \cdots = C_j = C$.

(Continued :)
Then $a < c < b$ and $f_c(c) = x$. Similarly, let

$$c' = 0.a_1 a_2 \cdots a_i C_{i+1} C_{i+2} \cdots C_j B x_1 x_2 \cdots x_n C y_1 y_2 \cdots_{13}.$$

Then we get $a < c' < b$ and $f_c(c') = -x$. Note that there exists a bijection between $x_1 x_2 \cdots x_n C y_1 y_2 \cdots_{13}$ of c, c' and $x = x_1 x_2 \cdots x_n . y_1 y_2 \cdots_{10}$. Thus $f_c[I] = \mathbb{R}$.

Yet, Conway's base 13 function is not a $(\star)$-function because it is defined only on $[0, 1)$. Hence we need to extend its domain to $\mathbb{R}$.

Theorem 4.6

Let $f_E : \mathbb{R} \to \mathbb{R}$ be a real function defined by

$$f_E(\pm x_1 x_2 x_3 \cdots x_n . y_1 y_2 y_3 \cdots {}_{13}) = f_c(0.y_1 y_2 y_3 \cdots {}_{13}).$$

Then f_E is a $(\star)$-function.

Proof. Let $n \in \mathbb{N} \cup \{0\}$ and let $x \in [0, 1)$ be defined as Definition 4.4. Then

$$f_E(n + x) = f_c(x).$$

Similarly, let $n, m \in \mathbb{N} \cup \{0\}$ $(n \leq m)$, $a, b \in [0, 1)$ $(a < b$ if $n = m)$ and let $I = (n + a, m + b)$ be any nonempty open interval of $\mathbb{R}$.
(i) Suppose $n = m$. Let $I' = (a, b)$. Then

$$f_E[I] = f_c[I'] = \mathbb{R}.$$

(Continued :)
(ii) Suppose $n < m$. Let $p_0 = n$ and $p_k = m$. Then there exists $p_1,\ p_2,\ \cdots, p_{k-1} \in \mathbb{N}$ such that $p_i - p_{i-1} = 1$ for all $1 \leq i \leq k$. Let $I_0 = (n + a, p_1),\ I_1 = [p_1, p_2),\ \cdots,\ I_{k-1} = [p_{k-1}, m),\ I_k = [m, m + b)$ and let
$I'_0 = (a, 1),\ I'_1 = [0, 1),\ \cdots,\ I'_{k-1} = [0, 1), I'_k = [0, b)$. Then

$$\begin{aligned} f_E[I] &= f_E[I_0] \cup f_E[I_1] \cup \cdots \cup f_E[I_{k-1}] \cup f_E[I_k] \\ &= f_c[I'_0] \cup f_c[I'_1] \cup \cdots \cup f_c[I'_{k-1}] \cup f_c[I'_k] \\ &= \mathbb{R}. \end{aligned}$$

Interval in $\mathbb{Z}^-$ can be considered similarly. Thus f_E is a $(\star)$-function.

Conway's base 13 function might not be that attractive, considering its complexity and unusual base 13 representation. However, it is interesting that the axiom of choice was not used to define this function. Moreover, this function gives us a specific detail of a $(\star)$-function. This can be compared to the Vitali set, which could only show existence of a $(\star)$-function. In this sense, Conway's base 13 function definitely has its own advantage compared to other ideas.