Finding a Jordan Canonical Form

Motivation

$T \in \mathcal{L}(V)$: linear Operator

- diagonal matrix
  - ↳ too simple
- triangular matrix
  - ↳ not simple enough

Motivation

- V : vector space over $\mathbb{F}$

  for $T \in \mathcal{L}(V)$, $p(x) \in \mathbb{F}[x]$

  $p(T)$ is well-defined

- Define $p(x)v := p(T)v$

  $\Rightarrow V$ as $\mathbb{F}[x]$-module for fixed $T \in \mathcal{L}(V)$

Def (Module)

R: ((commutative) ring (with identity)

R-module M is an abelian group with

$$R \times M \longrightarrow M$$
$$(r, v) \mapsto rv$$

$\Longleftrightarrow$ $\exists$ homomorphism of rings

$$\rho : R \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$$

Def.

$S \subseteq M$

- $\langle S \rangle = \{ r_1 v_1 + \cdots + r_n v_n \mid r_i \in R.\ v_i \in S\ \ n \geq 1 \}$
- S generates M if $\langle S \rangle = M$
- $\langle v \rangle = Rv$: cyclic submodule gen. by v
- M: finitely generated if $\langle S \rangle = M.\ |S| < \infty$
- $r_1 v_1 + \cdots + r_n v_n = 0 \Rightarrow r_i = 0\ \forall i$

  S : linearly independent

- v: torsion element of M

  if $rv = 0$ for some $0 \neq r \in R$

- torsion-free

  if module that has no nonzero torsion elements

- torsion module

  if all elements of M are torsion elements

- the annihilator of $v \in M$

$$\operatorname{ann}(v) = \{ r \in R \mid rv = 0 \}$$

- the annihilator of $N \subseteq M$

$$\operatorname{ann}(N) = \{ r \in R \mid rN = \{0\} \}$$

↳ ideals of R

Remark

$M = \langle v_1, \cdots, v_n \rangle$: torsion-module

i.e. $\exists\ a_i \in \operatorname{ann}(v_i)$

$\Rightarrow a = a_1 \cdots a_n$ annihilates every elements in M

i.e. $a \in \operatorname{ann}(M)$

Modules over a PID

let R be a PID

1) $r \in R$: irreducible $\Leftrightarrow$ (r) : maximal

2) $r \in R$: prime $\Leftrightarrow$ irreducible

3) R : UFD

4) R : Noetherian

$\Rightarrow$ M : n-gen. $\Rightarrow$ $\forall N \leq M$: n-gen.

Def

- any generator of $\mathrm{ann}(N)$ is called an order of N $=: o(N)$

Prop.

$$M = A \oplus B \Rightarrow o(M) = \mathrm{lcm}(o(A) . o(B))$$

Decomposition Theorems.

Thm 1. $M = M_{free} \oplus M_{tor}$

Thm 2 [The Primary Decomposition]

$ann(M) = \prod p_i^{e_i} \Rightarrow M = \bigoplus M_{p_i}$

where $M_{p_i} = \{ v \in M \mid p_i v = 0 \}$

Thm 3 [The Cyclic Decomposition]

$ann(M) = p^e \Rightarrow M = \bigoplus_{i=1}^{k} \langle v_i \rangle$. $ann(v_i) = (p^{e_i})$

where $e = e_1 \geq \cdots \geq e_k \geq 1$

such $k, e_1, \cdots, e_k$ are unique

V as $F[x]$-module, $T \in \mathcal{L}(V)$

$\Rightarrow F[x] \longrightarrow \mathrm{End}(V_T)$

$p(x) \longmapsto (v \mapsto p(x)v = p(T)v)$

$\dim V = n \Rightarrow \dim(\mathcal{L}(V)) = n^2$

$\therefore i, T, \cdots, T^{n^2}$ are linearly dependent

Hence $a_0 + a_1 T + \cdots + a_{n^2} T^{n^2} = 0$

not all a_i's zero

$\Rightarrow V_T$ is a torsion module

- $\mathbb{F}[x]$ is a PID

$\Rightarrow \operatorname{ann}(V_T) = \langle f(x) \rangle$

$f(x) = p_1(x)^{e_1} \cdots p_n(x)^{e_n}$

- $\operatorname{ann}(V_T) = \langle \underbrace{m_T(x)}_{} \rangle$

↳ minimal polynomial

- $m_T(x) = p_1^{e_1}(x) \cdots p_n^{e_n}(x)$

Primary Cyclic Decomposition

1) $V_T = V_{p_1} \oplus \cdots \oplus V_{p_n} \quad V_{p_i} = \{ v \in V \mid p_i^{e_i}(T)v = 0 \}$

2) $V_{p_i} = \langle v_{i,1} \rangle \oplus \cdots \oplus \langle v_{i,k_i} \rangle$

$$\Rightarrow V_T = [\langle v_{1,1} \rangle \oplus \cdots \oplus \langle v_{1,k_1} \rangle] \oplus \cdots$$

$$\oplus [\langle v_{n,1} \rangle \oplus \cdots \oplus \langle v_{n,k_n} \rangle]$$

- characteristic polynomial

$$C_T(x) = \prod_{i,j} p_i^{e_{i,j}}(x)$$

Thm

1) (Caley - Hamilton)

$$m_T \mid C_T \quad \text{i.e.} \quad C_T(T) = 0$$

2) $m_T = p_1^{e_{1,1}} \cdots p_n^{e_{n,1}}$

Rational Canonical Form

$V_T = \langle v \rangle$: T- cyclic

& $m_T(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$

& $\mathcal{B} = \{ v, Tv, \cdots, T^{n-1} v \}$

$\Rightarrow T(T^i v) = T^{i+1} v$

$T(T^{n-1} v) = T^n v = -a_0 v - \cdots - a_{n-1} T^{n-1} v$

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & \vdots & -a_1 \\ 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & & 1 & -a_{n-1} \end{pmatrix}$$: companion matrix of m_T

$=: C[m_T(x)]$

Rational Canonical Form

$$V_T = [\langle v_{1,1} \rangle \oplus \cdots \oplus \langle v_{1,k_1} \rangle] \oplus \cdots \oplus [\langle v_{n,1} \rangle \oplus \cdots \oplus \langle v_{n,k_n} \rangle]$$

For $\langle v_{i,j} \rangle$, $\mathcal{B}_{i,j} = \{ v_{i,j}, T v_{i,j}, \cdots, T^{e_{i,j}-1} v_{i,j} \}$

$$\mathcal{R} = \{ \mathcal{B}_{1,1}, \cdots, \mathcal{B}_{n,k_n} \}$$

$$\Rightarrow [T]_{\mathcal{R}} = \operatorname{diag}\left[C[p_1^{e_{1,1}}(x)], \cdots, C[p_n^{e_{n,k_n}}(x)] \right]$$

example

$T \in \mathcal{L}(\mathbb{R}^7)$ with $m_T(x) = (x-1)(x^2+1)^2$

1) $x-1, (x^2+1)^2, x^2+1$

2) $x-1, x-1, x-1, (x^2+1)^2$

1)
$$\left(\begin{array}{c|cccc|cc}
1 & & & & & & \\
\hline
 & 0 & 0 & 0 & -1 & & \\
 & 1 & 0 & 0 & 0 & & \\
 & 0 & 1 & 0 & -2 & & \\
 & 0 & 0 & 1 & 0 & & \\
\hline
 & & & & & 0 & -1 \\
 & & & & & 1 & 0
\end{array}\right)$$

2)
$$\left(\begin{array}{c|c|c|cccc}
1 & & & & & & \\
\hline
 & 1 & & & & & \\
\hline
 & & 1 & & & & \\
\hline
 & & & 0 & 0 & 0 & -1 \\
 & & & 1 & 0 & 0 & 0 \\
 & & & 0 & 1 & 0 & -2 \\
 & & & 0 & 0 & 1 & 0
\end{array}\right)$$

Jordan Canonical Form

- $m_T(x)$ splits over $\mathbb{F}$

  $\Rightarrow p_i^{e_{i,j}}(x) = (x-\lambda_i)^{e_{i,j}}$

- Since $\mathcal{B}_{i,j} = \{v_{i,j}, Tv_{i,j}, \cdots, T^{e_{i,j}-1}v_{i,j}\}$ is a basis,

  $\mathcal{C}_{i,j} = \{v_{i,j}, (T-\lambda_i)v_{i,j}, \cdots, (T-\lambda_i)^{e_{i,j}-1}v_{i,j}\}$ is a basis

$\Rightarrow T((T-\lambda_i)^{k-1}v_{i,j}) = (T-\lambda_i)^k v_{i,j} + \lambda_i v_{i,j}$

$$\Rightarrow [T]_{\mathcal{C}_{i,j}} = \begin{pmatrix} \lambda_i & 0 & & 0 \\ 1 & \lambda_i & & 0 \\ \vdots & 1 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_i \end{pmatrix} =: J(\lambda_i, e_{i,j})$$

Jordan block

Jordan Canonical Form

$$m_T(x) = (x-\lambda_1)^{e_1} \cdots (x-\lambda_n)^{e_n}$$

$\mathcal{C} = \bigcup \mathcal{C}_{i,j}$: Jordan basis

$$\Rightarrow [T]_{\mathcal{C}} = \mathrm{diag}[J(\lambda_1, e_{1,1}), \cdots, J(\lambda_n, e_{n,t_n}))$$

ex.

$$\left(\begin{array}{cc|c|c} \lambda & & & \\ 1 & \lambda & & \\ \hline & & \lambda & \\ \cline{3-4} & & & \lambda \end{array}\right)$$

#1

$$A = \begin{pmatrix} 3 & 5 & 1 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & -9 & -2 & 0 \\ -4 & -16 & -4 & -1 \end{pmatrix} \quad A - I = \begin{pmatrix} 2 & 5 & 1 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & -9 & -3 & 0 \\ -4 & -16 & -4 & -2 \end{pmatrix}$$

$$\chi_A(\lambda) = (\lambda - 1)^4$$

$$\Rightarrow m_A(x) =$$

#2

$$A - I = \begin{bmatrix} 2 & 5 & -2 & 5 & 6 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 & -2 & 0 & 0 \\ 0 & 4 & -1 & 3 & 6 & 1 & 0 \\ 0 & -2 & 1 & 0 & -2 & 0 & 0 \\ 0 & -4 & 1 & -9 & -12 & -3 & 0 \\ -4 & -24 & 6 & -16 & -27 & -4 & -2 \end{bmatrix}$$

$$\chi_A(x) = (x-1)^7$$

#3

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix}$$

#4

$$\begin{pmatrix} -2 & -7 & -1 & & \\ 1 & 3 & 0 & \multicolumn{2}{c}{0} \\ 0 & 1 & 2 & & \\ & & & -2 & -3 \\ & 0 & & 1 & 1 \end{pmatrix}$$

#5

$$\begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$$

$C_A(x) = (x-2)^2(x-3)$

$m_A(x) =$

#6

$$\begin{pmatrix} -6 & 1 & -6 & 11 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -2 \\ -4 & 0 & -4 & 7 \end{pmatrix}$$