Khovanov Homology

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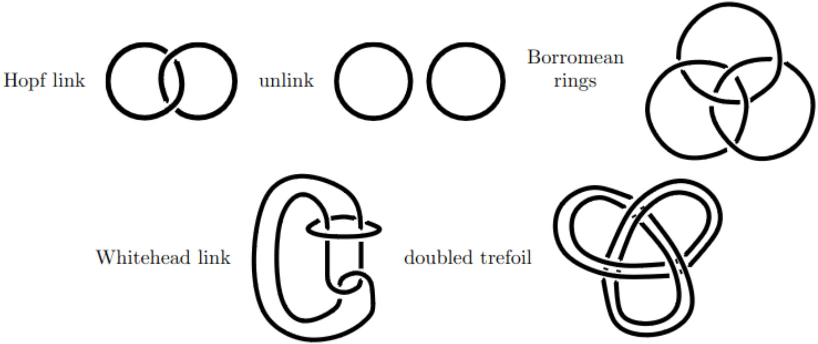
Knot and Link

Link

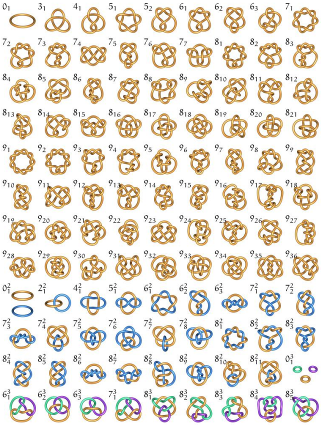
A link is simply a collection of (finitely-many) disjoint closed loops of string in $\mathbb{R}^3 \cup \{\infty\}$; each loop is called a component of the link.



Knot

A knot is a closed loop of string in $\mathbb{R}^3 \cup \{\infty\}$; two knots are equivalent if one can be wiggled around, stretched, tangled and untangled until it coincides with the other. Cutting and rejoining is not allowed.

Knot


0_1 3_1 4_1 5_1 5_2 6_1 6_2 6_3 7_1
7_2 7_3 7_4 7_5 7_6 7_7 8_1 8_2 8_3
8_4 8_5 8_6 8_7 8_8 8_9 8_{10} 8_{11} 8_{12}
8_{13} 8_{14} 8_{15} 8_{16} 8_{17} 8_{18} 8_{19} 8_{20} 8_{21}
9_1 9_2 9_3 9_4 9_5 9_6 9_7 9_8 9_9
9_{10} 9_{11} 9_{12} 9_{13} 9_{14} 9_{15} 9_{16} 9_{17} 9_{18}
9_{19} 9_{20} 9_{21} 9_{22} 9_{23} 9_{24} 9_{25} 9_{26} 9_{27}
9_{28} 9_{29} 9_{30} 9_{31} 9_{32} 9_{33} 9_{34} 9_{35} 9_{36}
0^2_1 2^2_1 4^2_1 5^2_1 6^2_1 6^2_2 6^2_3 7^2_1 7^2_2
7^2_3 7^2_4 7^2_5 7^2_6 7^2_7 7^2_8 8^2_1 8^2_2 8^2_3
8^2_4 8^2_5 8^2_6 8^2_7 8^2_8 8^2_9 8^2_{10} 8^2_{11} 0^3_1
6^3_1 6^3_2 6^3_3 7^3_1 8^3_1 8^3_2 8^3_3 8^3_4 8^3_5

Isotopic

Isotopic

Isotopy is a homotopy H such that for each fixed t, $H(x,t)$ gives an homeomorphism.

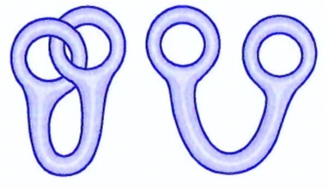


Figure: Isotopic

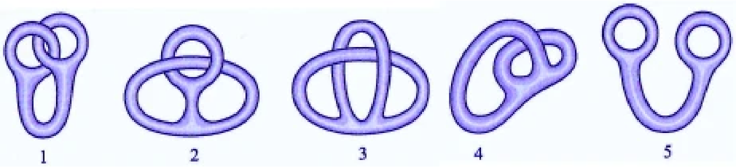
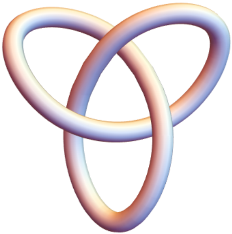


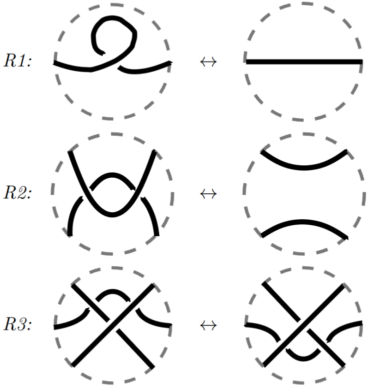
Figure: Isotopic

Diagram

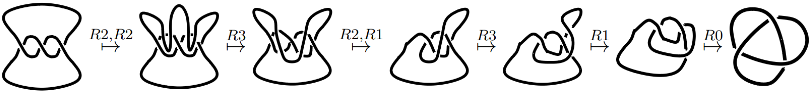
We have many different diagrams of the same knot!
Example)



Reidemeister moves

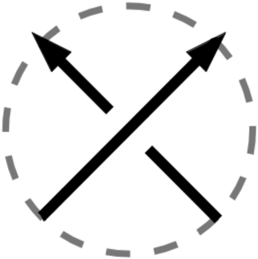


Reidemeister moves example

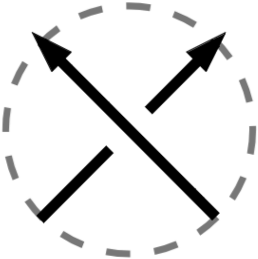


Orientation

We can give an orientation to the knot diagrams.



$+1$



-1

Let's define n_+, n_- for each one means the number of $\pm$ crossings.

Jones polynomial

We can compute the Jones polynomial with some rules for links K. First,

$$\langle \emptyset \rangle = 1$$

.

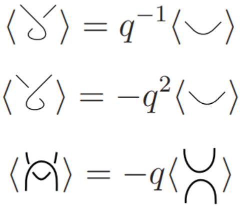
Smoothing

Let be the first one be the A-smoothing, and the second one be the B-smoothing.
Some authors write in 0 & 1 smoothing or 1 & x.

Skein Relation

$$\langle \bigcirc \rangle = q + q^{-1}$$

$$\langle K \bigcirc \rangle = (q + q^{-1})\langle K \rangle$$



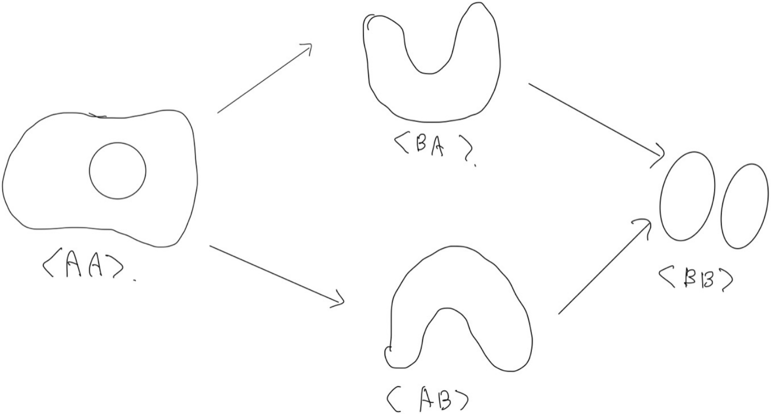
Knot category

Let $S(K)$ denote a category associated with the states of the bracket for a diagram K whose objects are the states, with sites labeled 0 and 1. In addition, a morphism in this category is an arrow from a state with a given number of $A's$ to a state with fewer $B's$[2].

Cube Category

Let $D^n = \{A, B\}^n$ be the n-cube category whose objects are the n-sequences from the set $\{A, B\}$ and whose morphisms are arrows from sequences with greater numbers of A's to sequence with fewer numbers of A's which is the poset category of subsets of $\{1, 2, \ldots, n\}$.

Hopf Link Smoothing



Categorification

Functor

Define the functor $R : D^n \to S(K)$ and $S : S(K) \to D^n$.

Object

Bracket state to the sequences in the cube category.

Morphism

Smoothing to sequences.

Then the two categories are equivalent.

Jones polynomial

For $\alpha \in \{A, B\}^n$, we will denote the associated smoothing by Γ_α.

Define

$$r_\alpha = \text{the number of B's in } \alpha$$

and

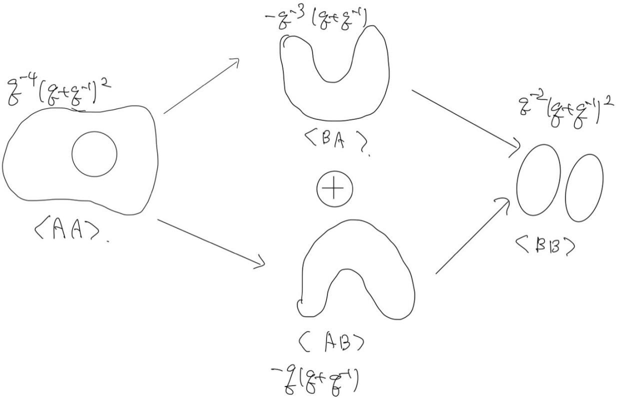
$$k_\alpha = \text{the number of circles in } \Gamma_\alpha.$$

Then

Jones Polynomial[5]

$$J(L) = \sum_{\alpha \in \{A,B\}^n} (-1)^{r_\alpha + n_-} q^{r_\alpha + n_+ - 2n_-} \left(q + q^{-1}\right)^{k_\alpha}.$$

Jones Polynomial of Hopf link



$$J(L) = q^{-6} + q^{-4} + q^{-2} + 1.$$

Conjectures

Unknotting problem

The unknot is the unique knot K with $V(K) = 1$.

Even though the Jones polynomial is the strong knot invariant, it cannot detect the unknot until now.

$D^n \to V$

Again the functor $F : D^n \to V$, where V is a vector space(abelian group).

Graded Vector Space

Graded Vector Space

Let $V = \mathbb{Q}\{1, x\}$(the $\mathbb{Q}$-vector space with basis 1 and x) and grade the two basis elements by $deg(1) = 1$ and $deg(x) = -1$.

Graded dimension

For a graded vector space $W = \oplus_m V^m$, defined the graded dimension

$$qdim(W) = \sum_m q^m \dim(W^m)$$

.

Grade Shifting

Graded shift

For a graded vector space W and an integer l we can define a new graded vector space $W\{l\}$ by

$$W\{l\}^m = W^{m-l}$$

and notice that $q\dim(W\{l\}) = q^l q\dim(W)$.

Properties

$$q\dim(W \otimes W') = q\dim(W)\, q\dim(W'),$$
$$q\dim(W \oplus W') = q\dim(W) + q\dim(W')$$

Cochain complex

To each $\alpha \in \{A, B\}^n$, associate the graded vector space

$$V_\alpha = V^{\otimes}\{r_\alpha + n_+ - 2n_-\}$$

and define

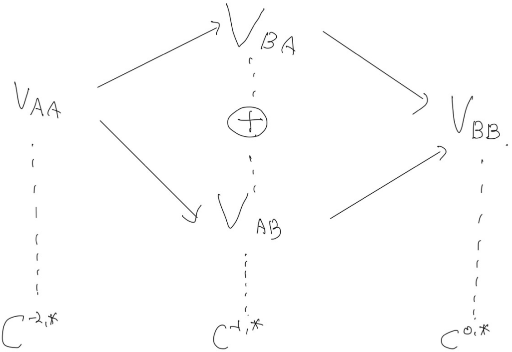
$$C^{i,*}(L) = \bigoplus_{\substack{\alpha \in \{A,B\}^n \\ r_\alpha = i + n_-}} V_\alpha .$$

Bi-grading

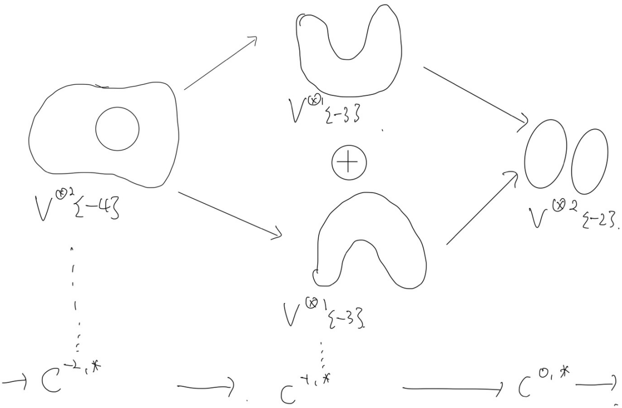
An element of $C^{i,j}(L)$ is said to have homological grading i and q-grading j. If $v \in V_\alpha \subset C^{*,*}(L)$,

$$\begin{aligned} i &= r_\alpha - n_- \\ j &= deg(v) + i + n_+ - n_- \end{aligned}$$

Vector Space



Vector Space



Linear map

Through the smoothing, two circles merge one circle, and one circle split to the one circle.
So we can define the linear map

$$\Delta : V \otimes V \to V$$
$$m : V \to V \otimes V.$$

such that

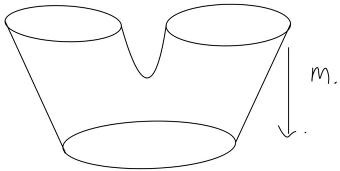
$$1 \otimes 1 \mapsto 1, \; 1 \otimes x, x \otimes 1 \mapsto x, \; x \otimes x \mapsto 0$$

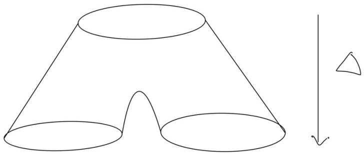
and

$$1 \mapsto 1 \otimes x + x \otimes 1, \; x \mapsto x \otimes x.$$

Cobordism

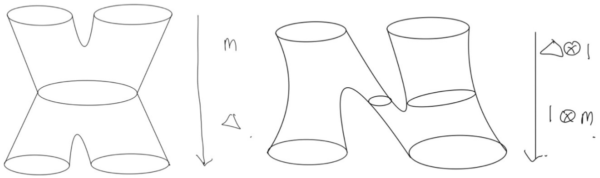
These maps can be represented by the surface cobordism.





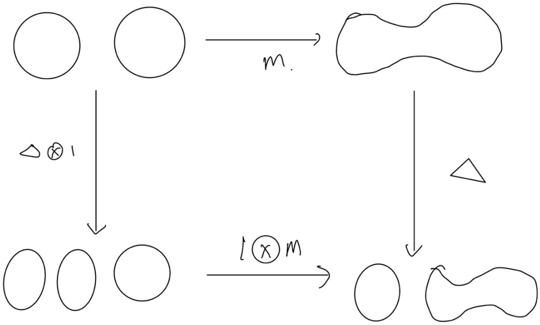
TQFT and Frobenius Algebra

These cobordism can represent the Frobenius Algebra through the 1+1 Topological Quantum Field Theory.



Boundary map

And it is equivalent to the commute digram.



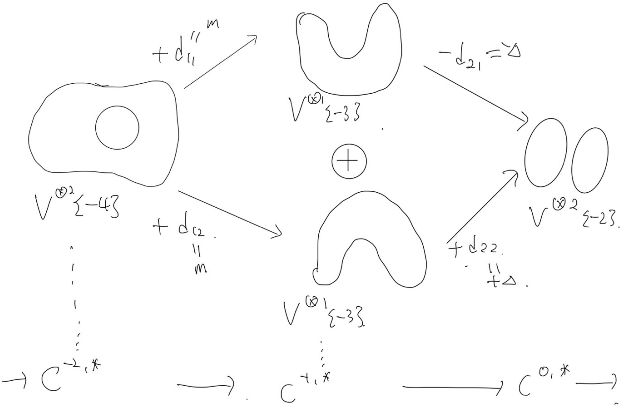
So that $d_1 d_2 = d_2 d_1$

Boundary map II

If we define the boundary map of the chain complex at slide 26 as

$$d^i(v) = \sum_{\substack{\xi \text{such that} \\ Tail(\xi)=\alpha}} \operatorname{sign}(\xi)\, d_\xi(v)$$

where $\operatorname{sign}(\xi) = (-1)^{\text{number of 1's to the left of } * \text{ in } \xi}$ and $v \in V_\alpha \subset C^{i,*}(L)$.
Then we can check that $d^{r+1} \circ d^r = 0$ from the previous slide.


+d11 = m
−d21 = Δ
V⊗2{−4}
V⊗1{−3}
+d12 = m
V⊗1{−3}
+d22 = +Δ
V⊗2{−2}
C−2,*
C−1,*
C0,*

Linear Map

Then we can define

d^1

$d^1 : V \otimes V \to V$ such that

$$d^1 (v_1 \otimes v_2) = (m (v_1 \otimes v_2), m (v_1 \otimes v_2))$$

.

d^2

$d^2 : V \to V \otimes V$ such that

$$d^2 (v_1, v_2) = -\Delta (v_1) + \Delta (v_2)$$

.

Linear map computation

Write the vector spaces with the ordered basis,

$$V \otimes V = \mathbb{Q}1 \otimes 1 \oplus \mathbb{Q}x \otimes 1 \oplus \mathbb{Q}1 \otimes x \oplus x \otimes x$$

and

$$V \oplus V = \mathbb{Q}^2 1 \oplus \mathbb{Q}^2 x$$

.

Linear map matrix

Then

$$d^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad d^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Homology

Easily we can compute the kernel and image of each matrices. So

Homological degree	-2	-1	0
Cycles	$\{1 \otimes x - x \otimes 1, x \otimes x\}$	$\{(1,1), (x,x)\}$	$\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$
Boundaries	-	$\{(1,1), (x,x)\}$	$\{1 \otimes x + x \otimes 1, x \otimes x\}$
Homology	$\{1 \otimes x - x \otimes 1, x \otimes x\}$	-	$\{1 \otimes 1, 1 \otimes x\}$
q-degrees	$-4, -6$		$0, -2$

Khovanov Homology

$i \backslash j$	0	-1	-2	-3	-4	-5	-6
-2					$\mathbb{Q}$		$\mathbb{Q}$
-1							
0	$\mathbb{Q}$		$\mathbb{Q}$				

Graded Euler Characteristic

Compute the graded Euler characteristic of this complex

$$\sum_i (-1)^i \, q\dim\left(C^{i,*}(L)\right) \in \mathbb{Q}\left[q^{\pm}\right]$$

is

$$q^{-6} + q^{-4} + q^{-2} + 1$$

that Jones polynomial of the Hopf link.

Significancy

Why we use the Khovanov Homology?

Unknot detector[4]

Khovanov homology can detect the unknot.

Mathematical Physics[7]

Related to the Gauge theory via Floer homology.

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