Model Predictive Control

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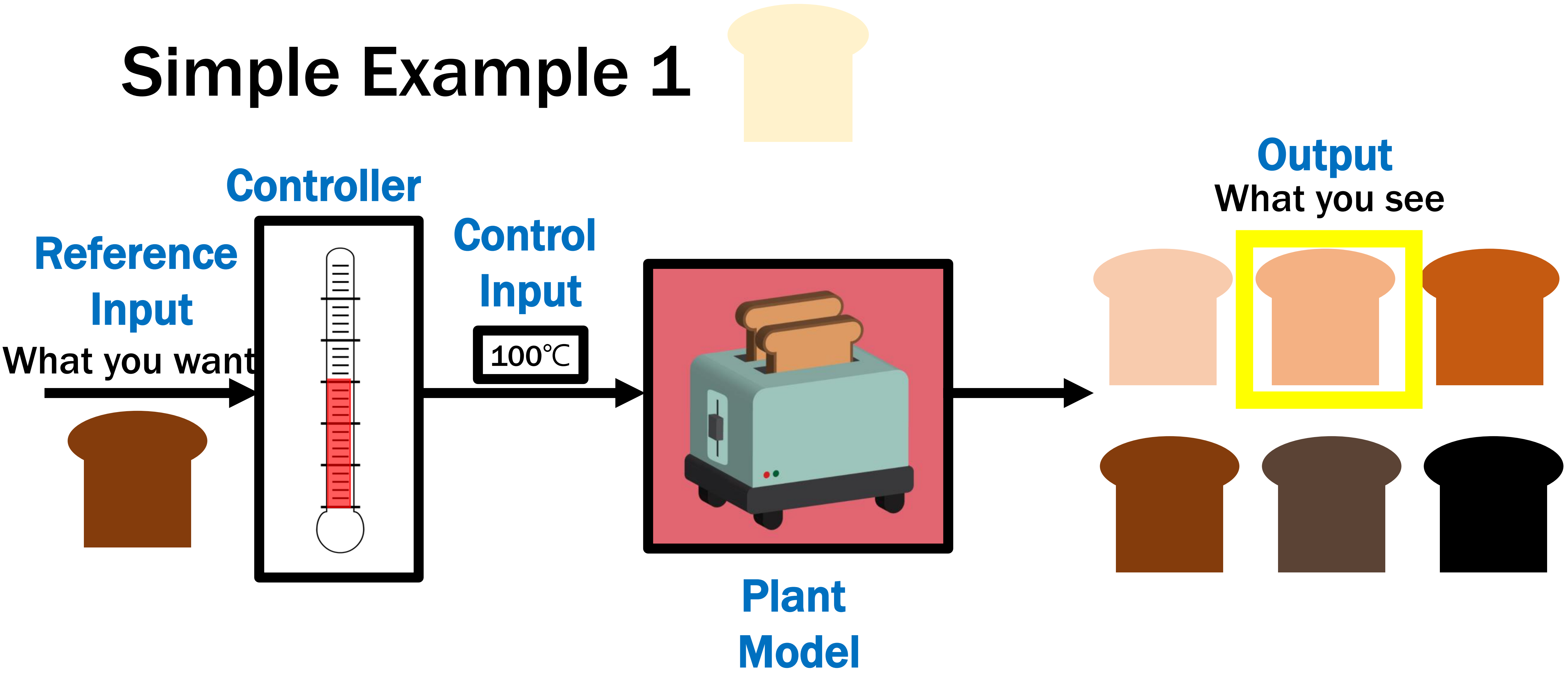
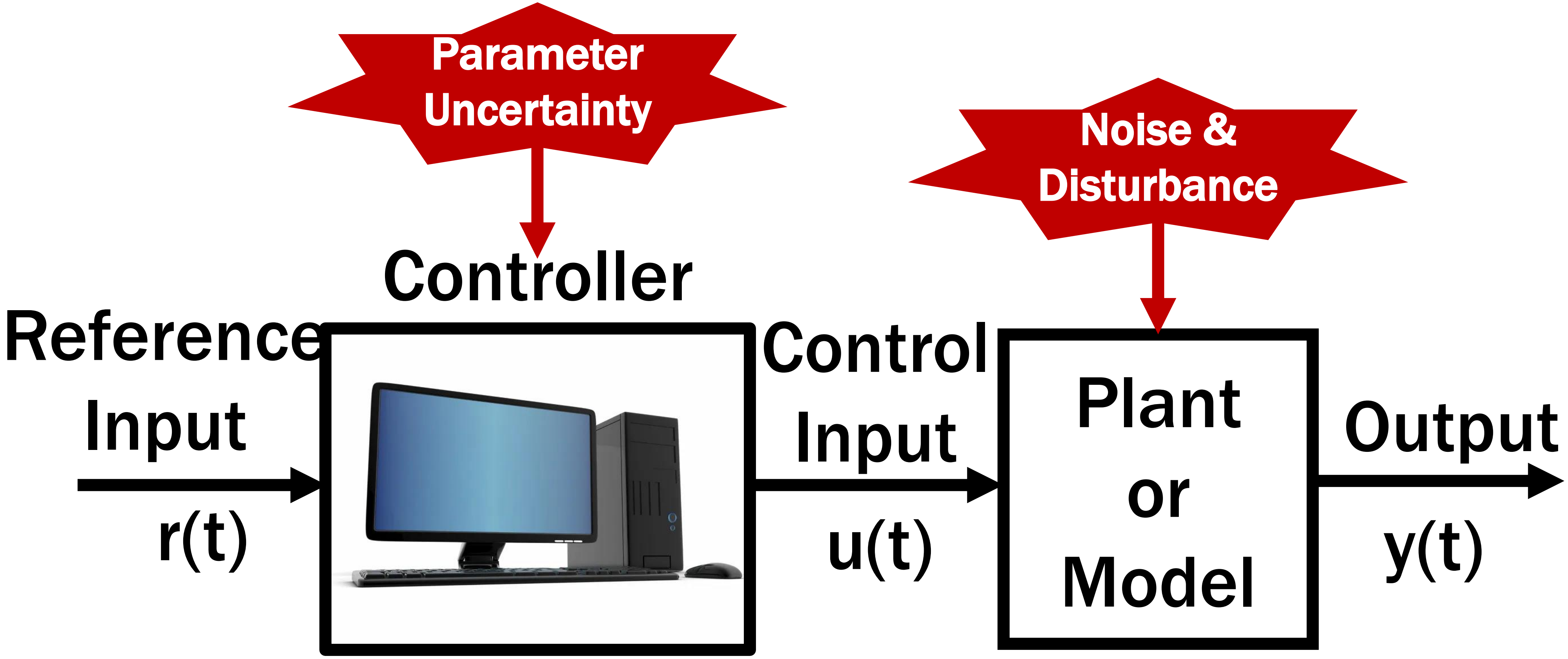
Part 1

Introduction to Optimal Control

Simple Example 1

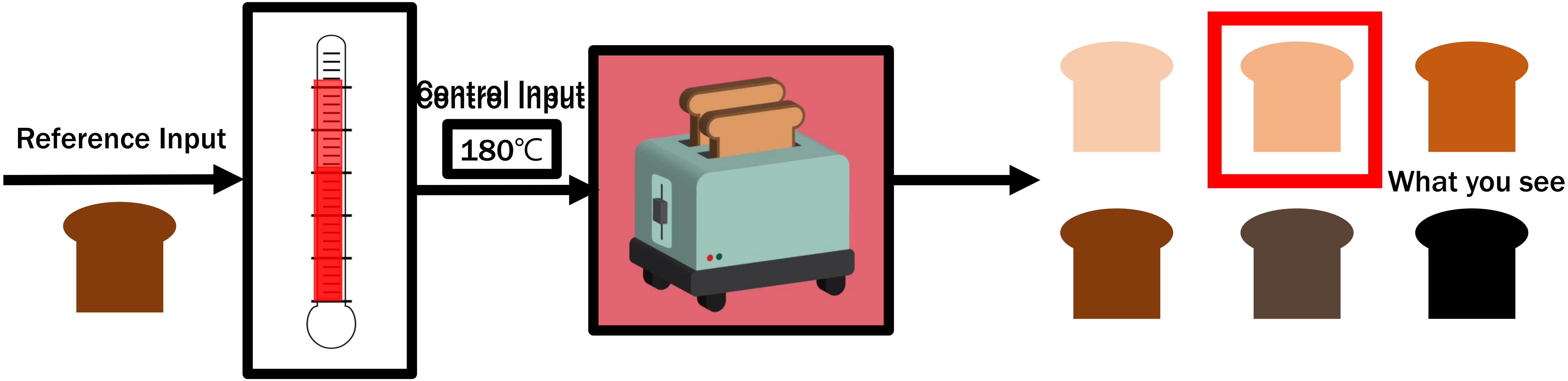


Open-loop System

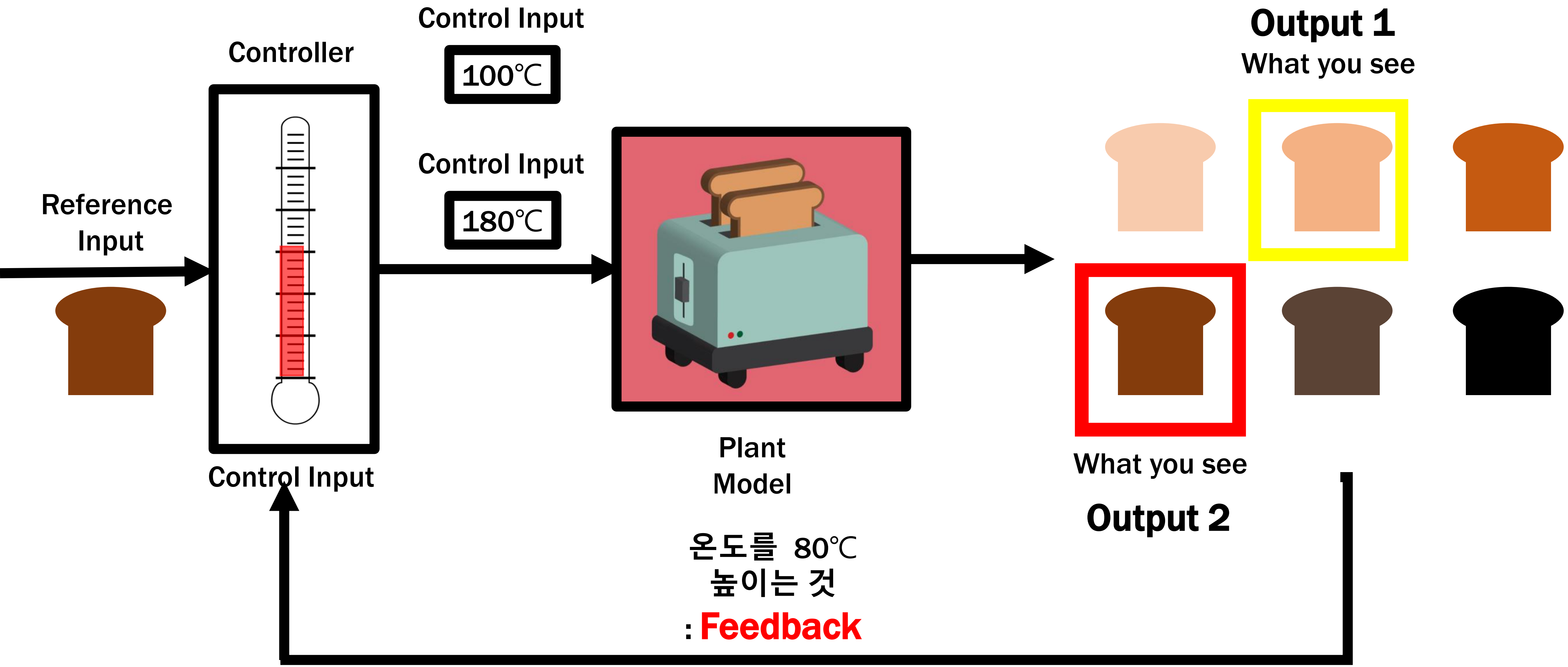


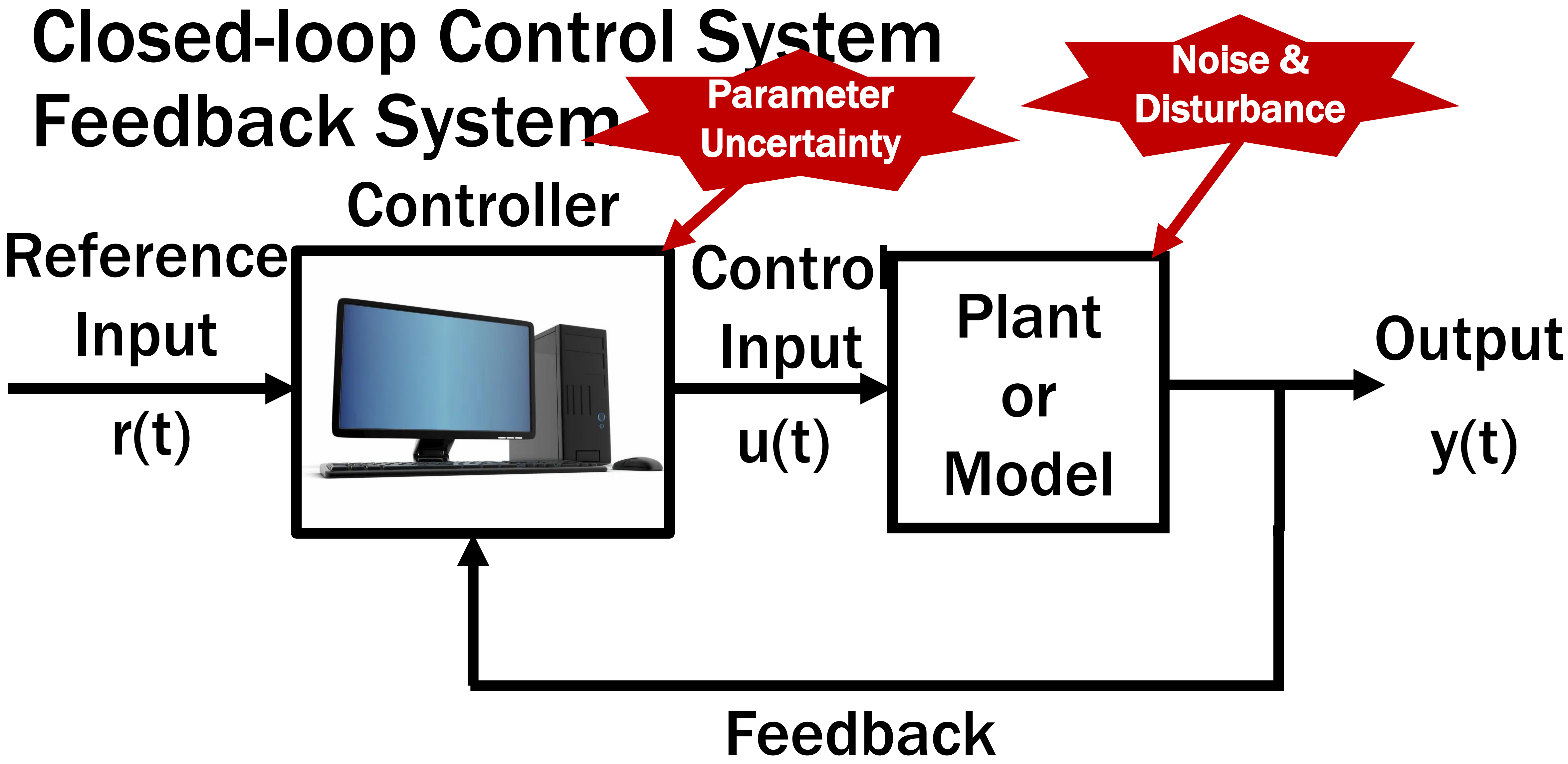
Objective : $r(t) \approx y(t)$

Simple Example 2



Simple Example 2




Closed-loop Control System
Feedback System
Parameter Uncertainty
Noise & Disturbance
Controller
Reference Input
r(t)
Control Input
u(t)
Plant or Model
Output
y(t)
Feedback

$$x(t+1) = f(\underbrace{x(t)}_{\textbf{\textit{state variable}}}, \underbrace{u(t)}_{\textbf{\textit{control input}}})$$

$$\underbrace{y(t)}_{\textbf{\textit{output}}} = g(\underbrace{x(t)}_{\textbf{\textit{state variable}}}, \underbrace{u(t)}_{\textbf{\textit{control input}}})$$

State Variable $x(t)$

시스템의 특성을 나타내는 변수

The state variables of a system consist of a minimum set of parameters which completely summarize the system's status in the following sense. If at any time t_0, the values of the state variables $x(t_0)$ are known, then the output $y(t_1)$ and the value $x(t_1)$ can be uniquely determined for any time $t_1,\ t_1 > t_0$, provided $u_{[t_0,t_1]}$ is known.

Output Feedback System

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

Feedback

Using $\boxed{\boldsymbol{u(t) = Fv(t) - Ky(t)}}$ gives

$$\begin{cases} \dot{x}(t) = \{A - BK[I + DK]^{-1}C\}x(t) + B\{F - K[I + DK]^{-1}DF\}v(t), \\ \boldsymbol{y(t) = [I + DK]^{-1}\{Cx(t) + DFv(t)\}}, \end{cases}$$

where K is feedback gain, F is feed-forward matrix and v(t) is external input.

Part 2

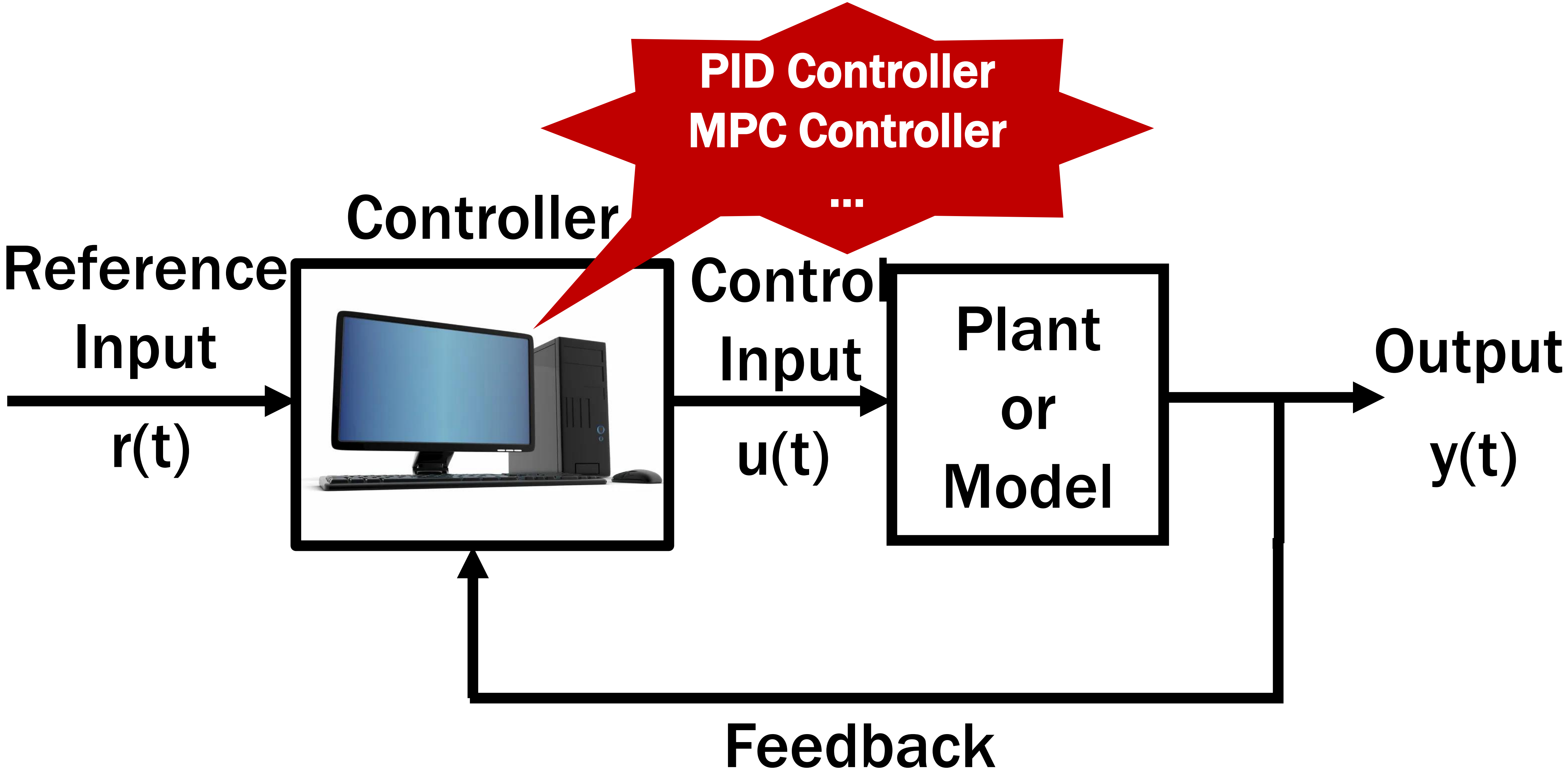
Model Predictive Control

미래에 대한 예측 X

미래에 대한 예측 O

TABLE 1 A list of the survey results in order of industry impact as perceived by the committee members.

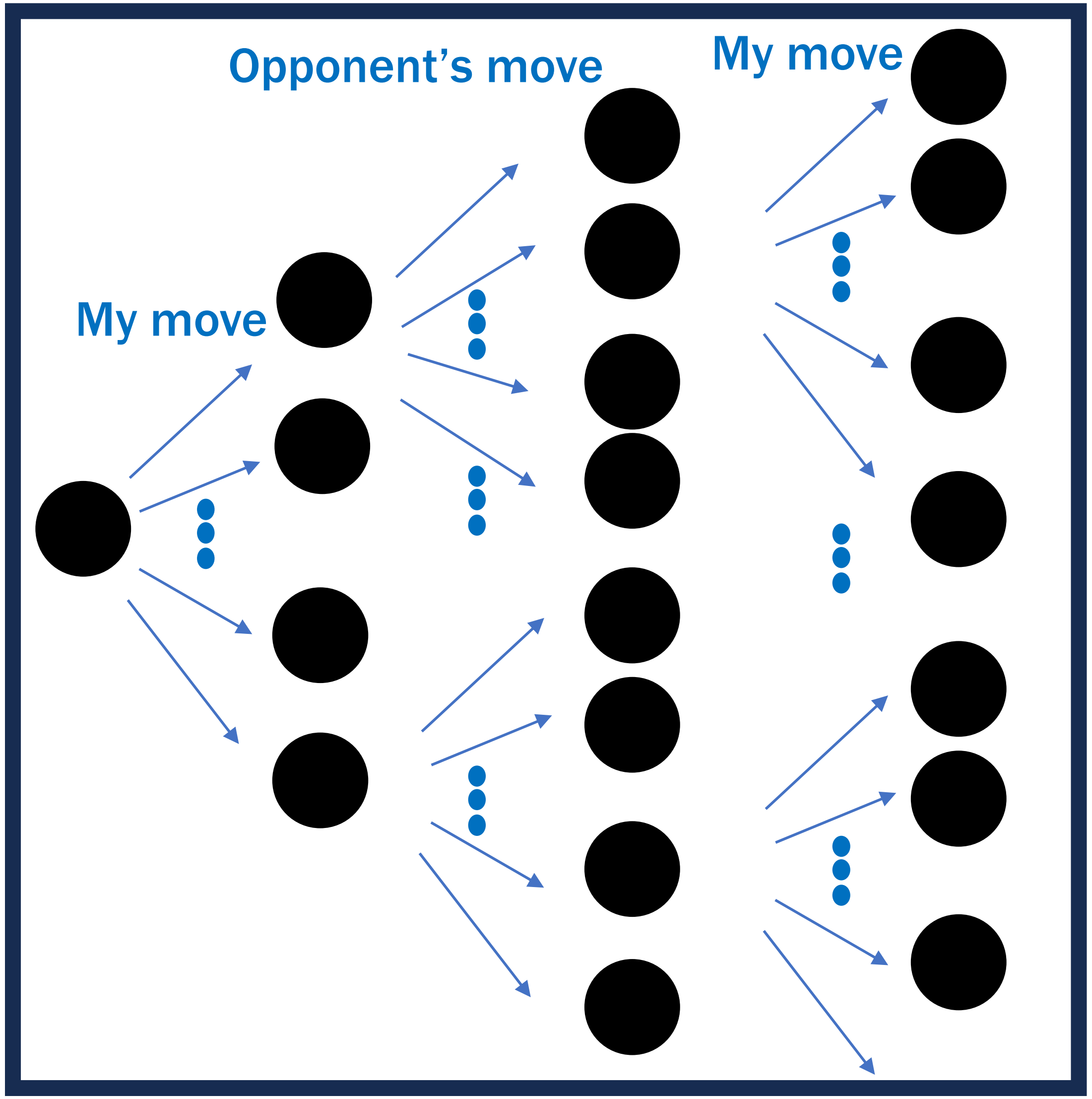
Rank and Technology	High-Impact Ratings	Low- or No-Impact Ratings
PID control	100%	0%
$\boxed{\text{Model predictive control}}$	78%	9%
System identification	61%	9%
Process data analytics	61%	17%
Soft sensing	52%	22%
Fault detection and identification	50%	18%
Decentralized and/or coordinated control	48%	30%
Intelligent control	35%	30%
Discrete-event systems	23%	32%
Nonlinear control	22%	35%
Adaptive control	17%	43%
Robust control	13%	43%
Hybrid dynamical systems	13%	43%

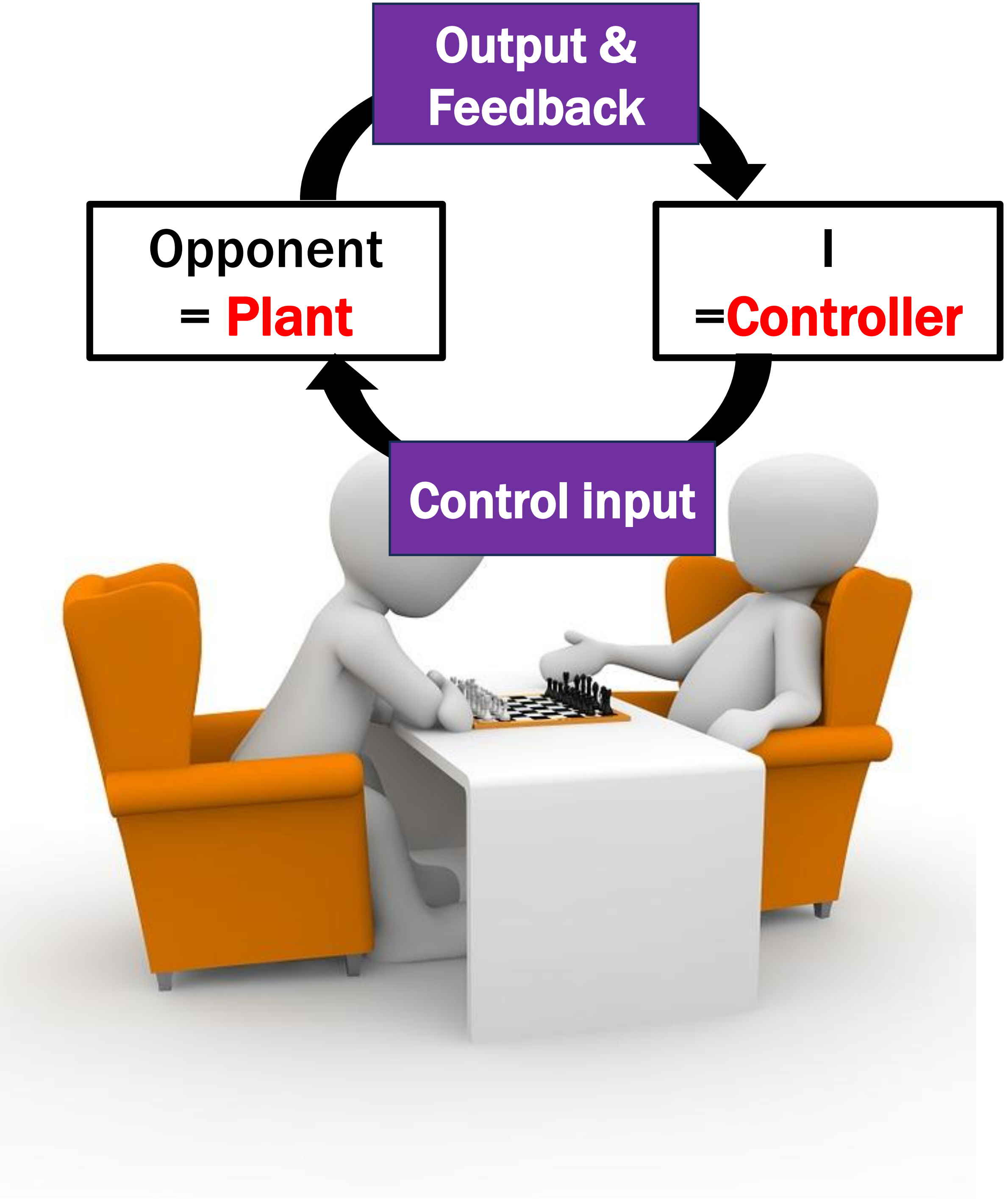

PID Controller
MPC Controller
...
Controller
Reference
Input
r(t)
Control
Input
u(t)
Plant
or
Model
Output
y(t)
Feedback

Part 2-1

Algorithm of MPC

Chess


Opponent's move
My move
My move


Output & Feedback
Opponent = Plant
I =Controller
Control input

Notation

- $u_{a|k}$: **t=k 에서 예측한 a 단계 이후의 control input 값**

ex) $u_{3|1}$ **: 현재 t=1에서 예측한 t=4에서의 control input 값**

- $u^*_{a|k}$**: t=k 에서 예측한 a 단계 이후의 최적의 control input 값**

ex) $u^*_{3|1}$**: t=1에서 예측한 t=4에서의 최적의 control input 값**

- $||\boldsymbol{x}||_{\boldsymbol{P}} = \sqrt{\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{P}\boldsymbol{x}}$

MPC Solver

Prediction time is <u>infinite</u>.
>>> Infinite-dimensional Optimization Problem

$$u^* = \underset{u}{argmin} \underbrace{\sum_{i=0}^{\infty} \left(||x_{i|k}||_Q^2 + ||u_{i|k}||_R^2 \right)}_{\text{Cost Function}},$$

$$\text{s.t. } \left.\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_t &= Cx_k \\ u_{min} \leq\ & u_{j|k} \leq u_{max}, \\ y_{min} \leq\ & y_t \leq y_{max}. \end{aligned}\right\} \text{Equality \& Inequality Constraints}$$

The Dual-Mode Prediction Paradigm

Prediction time is <u>finite</u>.
>>> Finite-dimensional optimization Problem

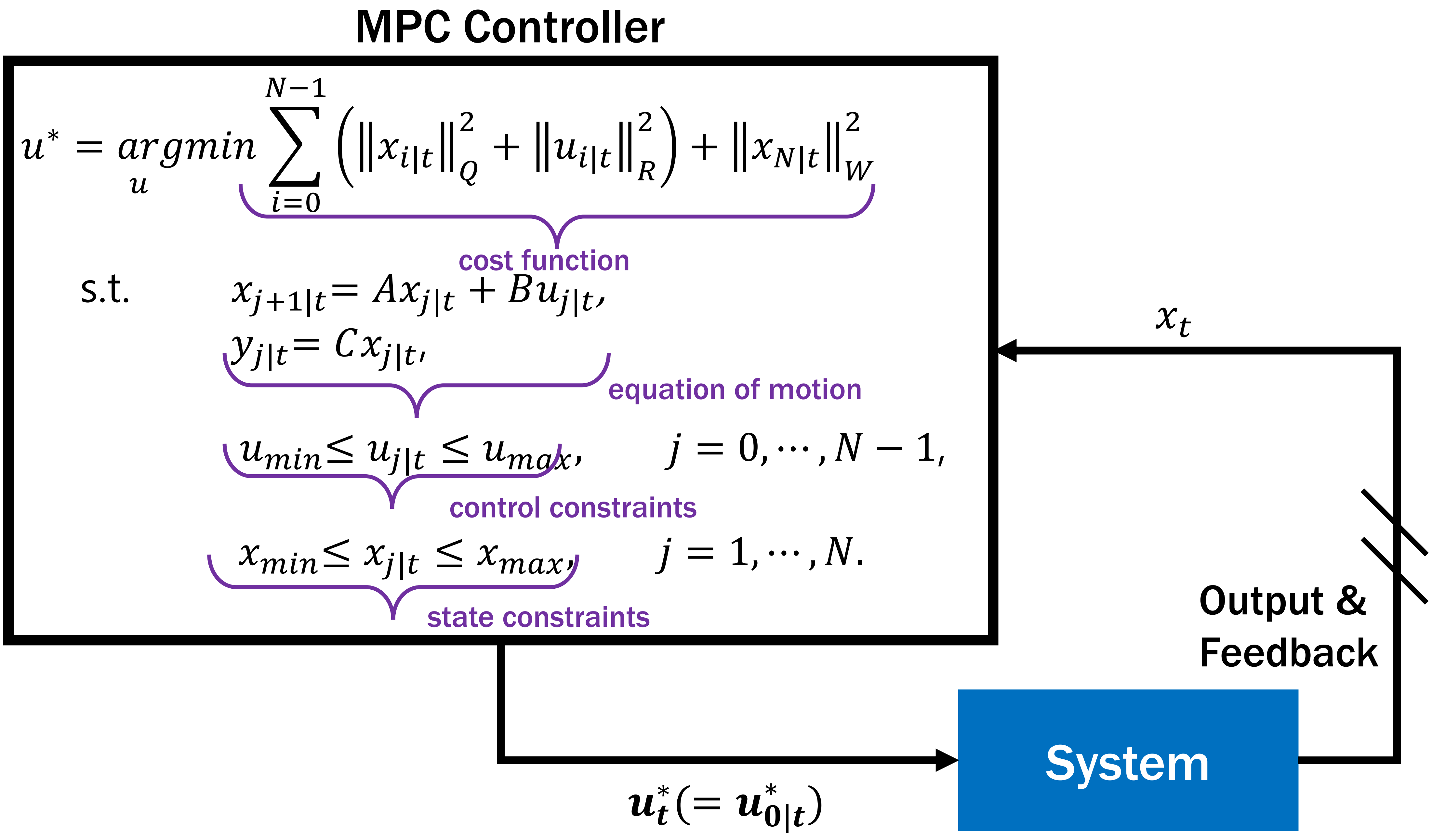
Prediction time is <u>infinite</u>.
>>> Infinite-dimensional Optimization Problem

$$\underset{u}{argmin}\, J(x_k, u_k) = \underset{u}{\operatorname{argmin}} \sum_{i=0}^{\infty} \|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2$$

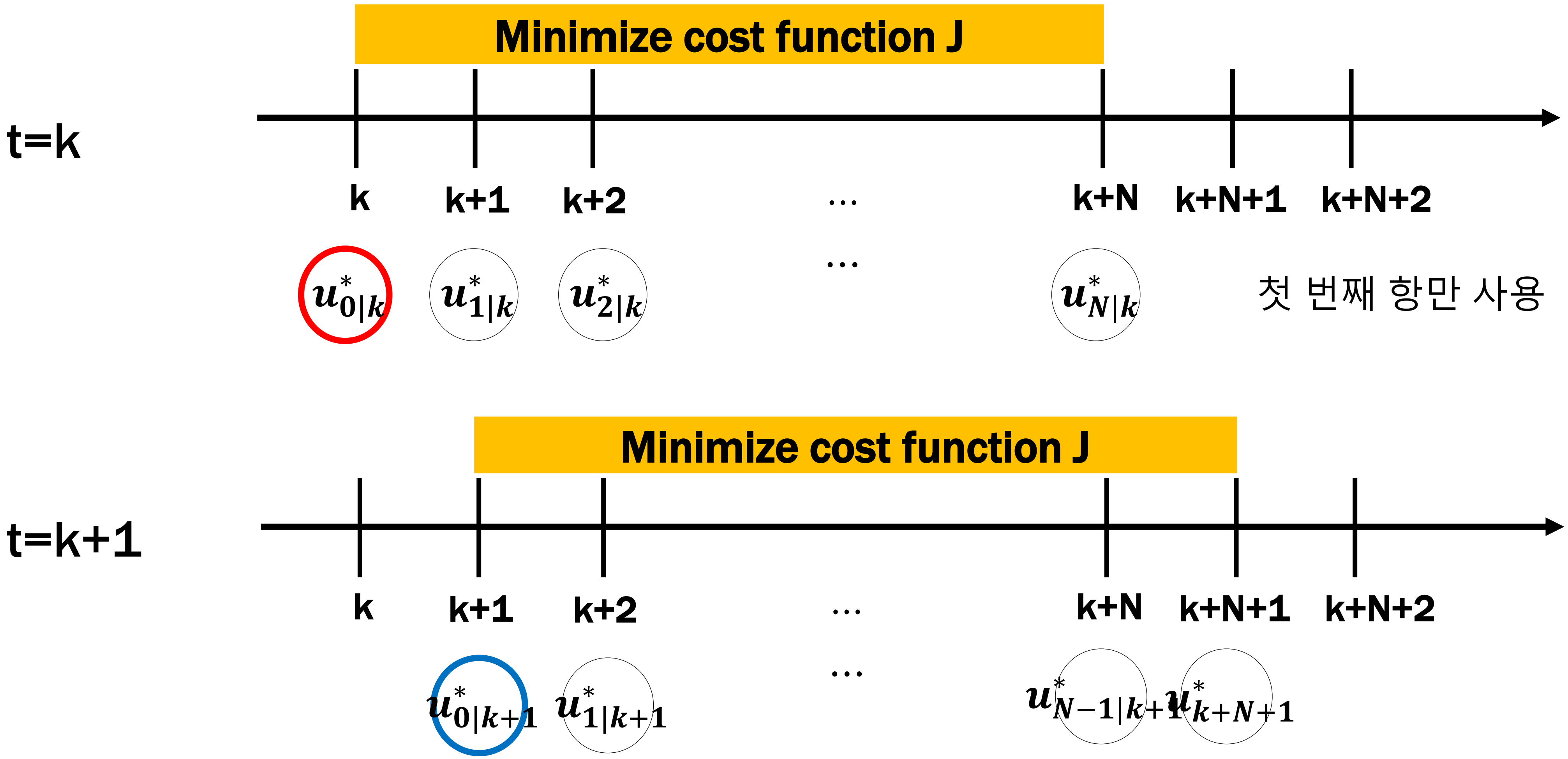
$$= \underset{\boldsymbol{u}}{\mathbf{argmin}} \underbrace{\overbrace{\sum_{i=0}^{N-1} \left(\|\boldsymbol{x}_{i|k}\|_Q^2 + \|\boldsymbol{u}_{i|k}\|_R^2 \right)}^{\text{Mode 1}}}_{\text{Running Cost}} + \underbrace{\overbrace{\|\boldsymbol{x}_{N|k}\|_{W'}^2}^{\text{Mode 2}}}_{\text{Terminal Cost}}$$

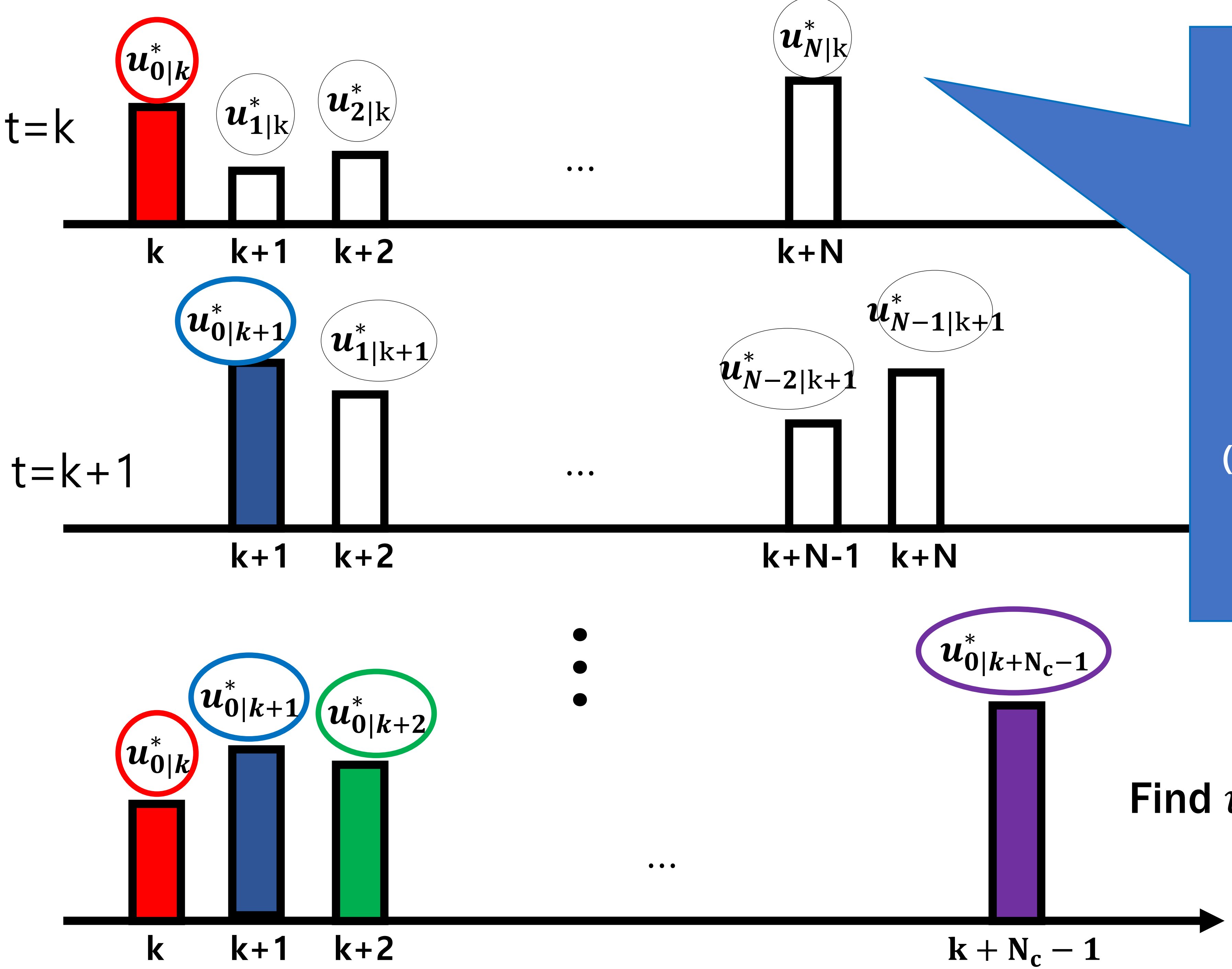
where W is the solution of the Lyapunov equation,

$$W = (A + BK)^T W (A + BK) + Q + K^T R K.$$


MPC Controller
$u^* = \underset{u}{argmin} \underbrace{\sum_{i=0}^{N-1} \left(\|x_{i|t}\|_Q^2 + \|u_{i|t}\|_R^2 \right) + \|x_{N|t}\|_W^2}_{\text{cost function}}$
s.t.
$\underbrace{x_{j+1|t} = Ax_{j|t} + Bu_{j|t}, \quad y_{j|t} = Cx_{j|t},}_{\text{equation of motion}}$
$\underbrace{u_{min} \leq u_{j|t} \leq u_{max},}_{\text{control constraints}} \quad j = 0, \cdots, N-1,$
$\underbrace{x_{min} \leq x_{j|t} \leq x_{max},}_{\text{state constraints}} \quad j = 1, \cdots, N.$
x_t
Output & Feedback
System
$\boldsymbol{u_t^*} (= \boldsymbol{u_{0|t}^*})$

Model Predictive Control
=Receding Horizon(이동 시간 구간) Control





첫 번째 항만 사용하고

모두 버리는 이유

: control input 값을 넣었을때

어떤 output 값이 나올지 모름

(parameter uncertainty, noise,

disturbance)

Find $u^* = \{u^*_{0|0}, u^*_{0|1}, u^*_{0|2}, \cdots\}$.

Part 2-2

Other Types of MPC

Other Types of MPC

- Linear Quadratic MPC
- Nonlinear MPC
- Decentralized MPC
- Stochastic MPC
- Robust MPC
- Economic MPC
- Adaptive MPC
- Hybrid MPC
- …

Robust MPC

System의 noise & disturbance에 잘 대처함

$$\min_u J(x_k, u_k) = \sum_{i=0}^{\infty} \left(||x_{i|k}||_Q^2 + ||u_{i|k}||_R^2 \right),$$

s.t. $\boldsymbol{x_{j+1|k} = Ax_{j|k} + Bu_{j|k}}$,

$\boldsymbol{y_{j|k} = Cx_{j|k}}$

$u_{min} \le u_{j|k} \le u_{max}$,

$y_{min} \le y_{j|k} \le y_{max}$.

$$\boxed{\boldsymbol{x_{j+1|k} = Ax_{j|k} + Bu_{j|k} + w_{j|k}}}$$

$$\boxed{\boldsymbol{x_{k+1} = A(\delta_{j|k})x_{j|k} + B(\delta_{j|k})u_k + w_{j|k}}}$$

$$\boxed{\boldsymbol{y_{j|k} = Cx_{j|k} + v_{j|k}}}$$

$$\boxed{\boldsymbol{y_{j|k} = C(\delta_{j|k})x_{j|k} + v_{j|k}}}$$

Robust MPC

$$\min_{u} R^{cost}(J(x_k, u, \delta))$$

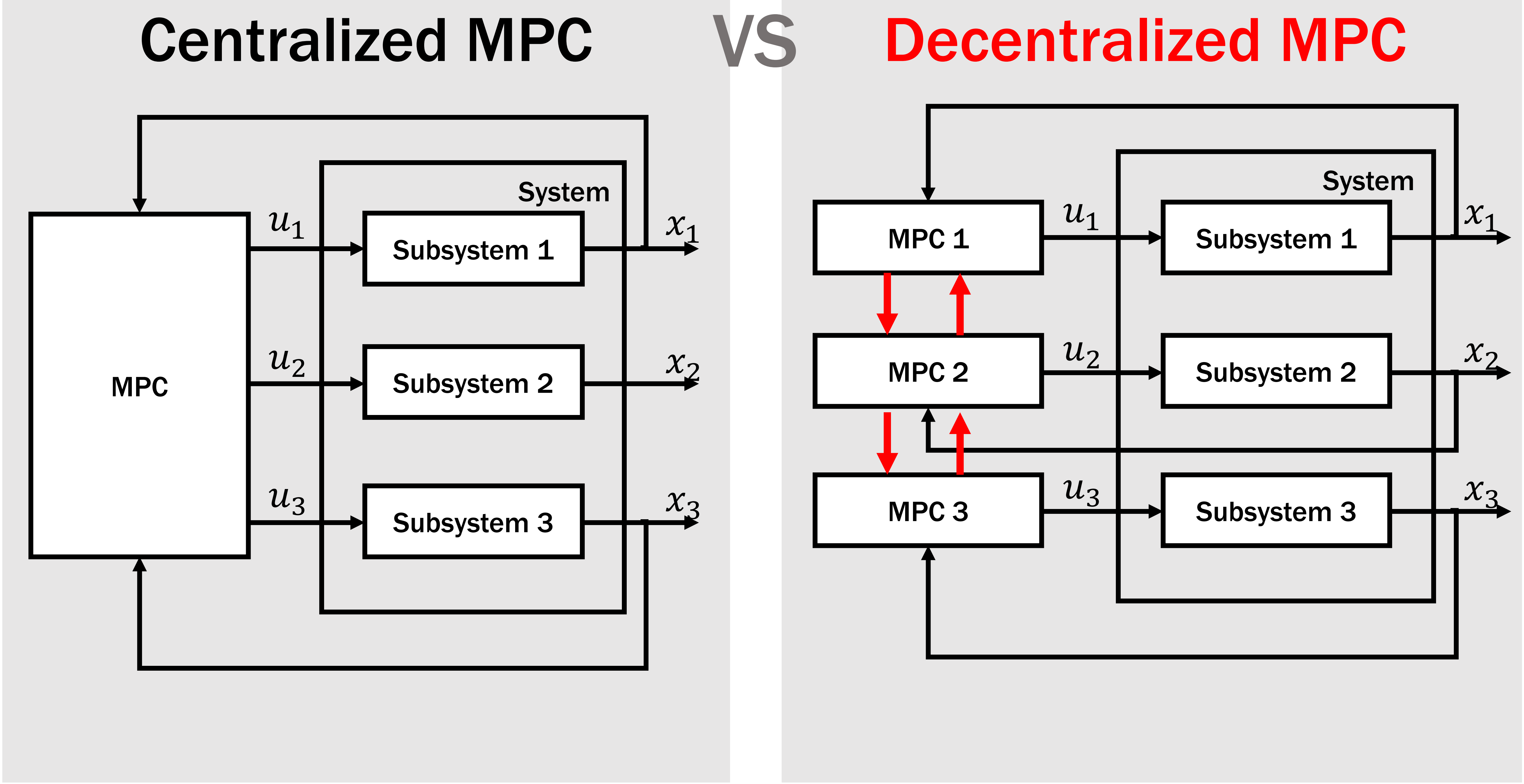
s.t $\boldsymbol{x_{j+1|k}} = \boldsymbol{A}\left(\boldsymbol{\delta_{j|k}}\right)\boldsymbol{x_{j|k}} + \boldsymbol{B}\left(\boldsymbol{\delta_{j|k}}\right)\boldsymbol{u_{j|k}} + \boldsymbol{F}\boldsymbol{w_{j|k}},$

$$\boldsymbol{y_{j|k}} = \boldsymbol{C}\left(\boldsymbol{\delta_{j|k}}\right)\boldsymbol{x_{j|k}} + \boldsymbol{D}\left(\boldsymbol{\delta_{j|k}}\right)\boldsymbol{u_{j|k}} + \boldsymbol{v_{j|k}},$$

$$R^{const}\left(c_{ij}(x_{j|k}, u_{j|k}, \delta_{j|k}, y_{j|k})\right) \leq 0,$$

$$j \in \mathbb{Z}_{[0,N_p-1]},\ i \in \mathbb{Z}_{\left[1,N_c^j\right]},\ x_{0|k} = x_k$$

Decentralize : 분산시키다



Decentralized MPC

$$\min_{u} \sum_i \sum_{j=0}^{N-1} \left[|| \, x_i(k+j) ||_{Q_i}^2 + || \, u_i(k+j) ||_{R_i}^2 \right] + || \, x_i(k+N) ||_{P_i}^2,$$

s.t. $x_{k+1} = Ax_k + Bu_k,$

$x \in X = \prod_i X_i \, , \; u \in U = \prod_i U_i \, .$

For $j = 0, \; \cdots, N-1,$

$u_i(k+j) \in U_i,$

$u_l(k+j) = u_l(k+j)^{c-1}, \; \forall l \neq i,$

$x(k+j) \in X,$

$x(k+N) \in X_f.$

Thank you