What is Topology and ‘Compactness'?

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Index

1. What is Topology?

- Definition of Topology

Definition. A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

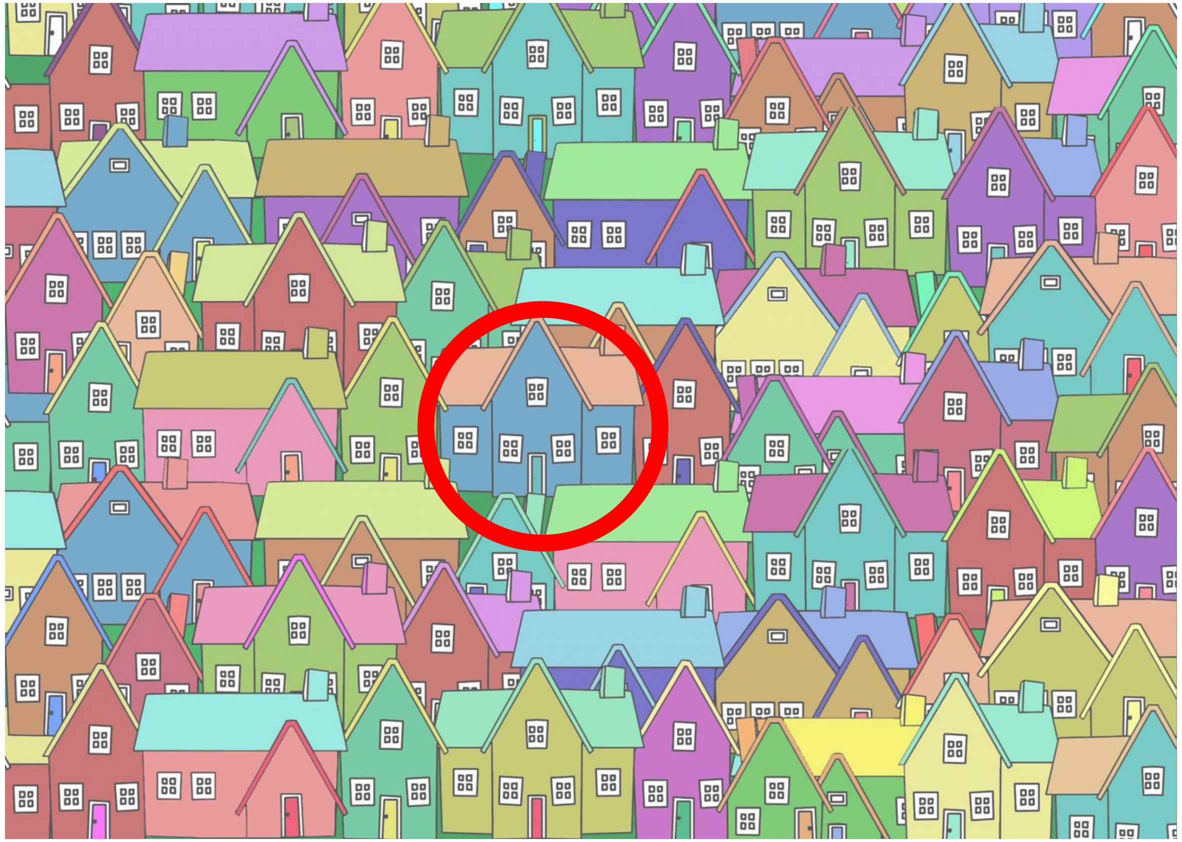
(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Simple Explanation about Topology

Define ‘Neighborhood’ !



How?

1. By distance
   ex) Within 1km radius

2. Only acquaintance

3. Those following each other in Instagram

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So many ways!

Simple Explanation about Topology

Example about Topology

What is an open set in $\mathbb{R}$?

Example about Topology

What is an open set in $\mathbb{R}^2$?

Interior, Closure, Boundary

Simple Explanation about Topology

In short, Topology (on a set) determines what an open set(neighborhood) is.

More specifically, it determines open, closed, interior, closure, boundary.

2. What is Compactness?

- Definition of Compactness

Definition. A subset K of a topological(metric) space is called *'compact'* if every open cover of K has a finite subcover.

Let's unpack it!

Open cover, Subcover?

In short, Topology (on a set) determine

2. Compact sets

Definition. A subset K of a topological(metric) space is called *'compact'* if every open cover of K has a finite subcover.

Is [0,1] indeed a compact set?

Sequential compactness

Definition. A set K is *(sequentially) compact* if and only if every sequence in K has a con~~vergent subsequence~~ in K.

What do sequentially compact sets look like?

What do sequentially compact sets look like?

sequentially compact sets

Sequentially compact sets $\Longleftrightarrow$ Bounded & Complete

Does it end?

No, we have to consider the infinite-dimensional space.

Infinite-dimensional space?

Consider a sequence $\{a_n\} \in \mathbb{R}^\infty$.

$$a_1 = (1,0,0,0, \ldots)$$
$$a_2 = (0,1,0,0, \ldots)$$
$$a_3 = (0,0,1,0, \ldots)$$
$$a_n = (0,0,0,0, \ldots, 1,0,0, \ldots)$$

This sequence is bounded but not convergent.

=> Should be totally bounded.

What is total boundedness?

Definition. A set K of a metric space is *totally bounded* if and only if for any given fixed $\varepsilon > 0$, K can be covered by finite many balls or radius ε.

Then... What do compact sets look like?

compact sets $\Longleftrightarrow$ Totally bounded & Complete

$\Updownarrow$

It is small enough, fully filled enough, and no escapeway for sequences to escape out.

3. Why is it important?

- Extreme Value Theorem

Theorem. If a real-valued function f is continuous in a closed interval [a,b], then f must attain a maximum and minimum.

- (Wick) Extreme Value Theorem

Theorem. If a real-valued function f is continuous in a closed interval [a,b], then f is bounded.

3. Extreme Value Theorem

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3. Extreme Value Theorem

3. Property of compact sets

Local

Global