11.27. MIMIC Seminar.

⟨ Differential Forms & de Rham Cohomology ⟩.

1. Review on Calculus.

- (Fundamental theorem of Line integral)

  $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $\gamma : [a.b] \to U$ smooth.

  $$\Rightarrow \quad \int_\gamma \operatorname{grad} f \cdot ds = f(\gamma(b)) - f(\gamma(a)) = \int_{\partial[a.b]} f.$$

- (Green's theorem)

  $D \subseteq \mathbb{R}^2$, $\vec{F}$: smooth v.f on D

  $$\int_{\partial D} \vec{F} \cdot ds = \iint_D \operatorname{rot} \vec{F} \, dA$$

- (Divergence Theorem)

  $R \subset \mathbb{R}^3$, $\vec{F}$: smooth v.f on R,

  $$\iint_{\partial R} \vec{F} \cdot dS = \iiint_R \operatorname{div} \vec{F} \, dV.$$

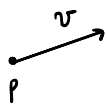
- (Stokes' theorem)

  $\vec{F}$: smooth v.f on a surface $S \subseteq \mathbb{R}^3$.

  $$\iint_S \operatorname{curl} \vec{F} \cdot dS = \int_{\partial S} \vec{F} \cdot ds.$$

* Geometric Tangent Spaces.

$$\mathbb{R}^n_p := \text{the tangent space of } \mathbb{R}^n \text{ at } p.$$
$$= \{p\} \times \mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}.$$



$\rightsquigarrow$ We abbreviate (p, v) as v_p.

$T_p\mathbb{R}^n$ forms a real vector space under the canonical operations :

$$v_p + w_p = (v+w)_p, \quad c \cdot v_p = (cv)_p.$$

For a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we define the directional derivative in the direction v at p by

$$D_v f(p) := \left.\frac{d}{dt}\right|_{t=0} f(p+tv)$$

$\Rightarrow$ We use the notation $D_v|_p f$ instead of $D_v f(p)$

$\Rightarrow$ $D_v|_p f$ can be considered as

$$D_v|_p : C^\infty(\mathbb{R}^n) \to \mathbb{R}. \quad \text{(called a derivation)}$$

satisfying the following properties :

- $D_v|_p$ is linear.
- $D_v|_p(fg) = D_v|_p(f)\, g(p) + f(p) D_v|_p(g)$. (Leibniz Rule)

Let $T_p\mathbb{R}^n = \{$ all derivations of $C^\infty(\mathbb{R}^n)$ at $p\}$.

$\Rightarrow$ $T_p\mathbb{R}^n$ forms a vector space under the operations :

$$(v+w)f = vf + wf \quad , \quad (cv)f = c(vf)$$

for $f \in C^\infty(\mathbb{R}^n)$

One can check $\begin{array}{ccc} \mathbb{R}^n_p & \longrightarrow & T_p\mathbb{R}^n \\ v_p & \longmapsto & D_v|_p \end{array}$ is an isomorphism.

$\{$Tangent vectors on $\mathbb{R}^n\}$ = $\{$ Derivations on $C^\infty(\mathbb{R}^n)\}$.

In this reason,

$$D_{e_i}|_p f = \lim_{t\to 0} \frac{f(p+te_i)-f(p)}{t} = \frac{\partial f}{\partial x_i}(p) := \frac{\partial}{\partial x_i}\Big|_p f$$

$\rightsquigarrow$ $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}$ is a basis for $T_p\mathbb{R}^n$.

$\rightsquigarrow$ For fixed $f \in C^\infty(\mathbb{R}^n)$,

$D_v|_p f = df_p(v)$ $\rightsquigarrow$ $df_p : T_p\mathbb{R}^n \to \mathbb{R}$ linear.

$\Rightarrow$ $df_p \in (T_p\mathbb{R}^n)^* = T_p^*\mathbb{R}^n$. : the cotangent space of $\mathbb{R}^n$ at p.

$\rightsquigarrow$ $\{dx_1|_p, \cdots, dx_n|_p\}$ is a basis for $T_p^*\mathbb{R}^n$.

$\uparrow$ dual basis of $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p \right\}$.

2. Differential Forms on $\mathbb{R}^n$.

$\mathbb{R}$-v.s + ring structure.

Define Ω^* to be the $\mathbb{R}$-algebra generated by $dx_1, \cdots, dx_n$ where $x_1, \cdots, x_n$ are the standard linear coordinates on $\mathbb{R}^n$ with the relation

- $dx_i dx_i = 0 \quad \forall i = 1, 2, \cdots, n.$
- $dx_i dx_j = -dx_j dx_i \quad \forall i \neq j$

$\rightsquigarrow$ Ω^* has a basis

$$1, \ dx_i, \ \underset{i<j}{dx_i dx_j}, \ \underset{i<j<k}{dx_i dx_j dx_k}, \ \cdots, \ dx_1 \cdots dx_n$$

The (smooth) differential forms on $\mathbb{R}^n$ are elements of

$$\Omega^*(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Omega^*$$

(i.e. if $\omega \in \Omega^*(\mathbb{R}^n)$, ω can be uniquely written as

$$\omega = \sum_{i_1 < \cdots < i_p} f_{i_1 i_2 \cdots i_p} dx_{i_1} \cdots dx_{i_p} = \sum f_I dx_I. \rightsquigarrow p\text{-form}$$

$$\Rightarrow \Omega^*(\mathbb{R}^n) = \bigoplus_{p=0}^{n} \Omega^p(\mathbb{R}^n) \quad \text{where } \Omega^p(\mathbb{R}^n) = \{p\text{-forms on } \mathbb{R}^n\}.$$

is graded $\mathbb{R}$-algebra.

3. Exterior Derivatives & de Rham Complex.

We have seen that for $f \in C^\infty(\mathbb{R}^n)$,

$$df_p : T_p\mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\begin{array}{ccc} v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\Big|_p & \longmapsto & \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(p) \\ \cong & & \| \\ (v_1, \cdots, v_n) & \longmapsto & v \cdot \operatorname{grad} f(p) \end{array}$$

$\Rightarrow$ Since $dx_i|_p(v) = dx_i|_p\left(\sum_{j=1}^{n} v_j \frac{\partial}{\partial x_j}\Big|_p\right) = \sum_{j=1}^{n} v_j \delta_{ij} = v_i$

We can write

$$df_p(v) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i|_p(v)$$

$$\Rightarrow df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

$\rightsquigarrow$ Generalization : There is a differential operator

$$d : \Omega^p(\mathbb{R}^n) \longrightarrow \Omega^{p+1}(\mathbb{R}^n)$$

defined as follows :

① if $f \in \Omega^0(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)$, $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$.

② if $\omega = \sum f_I dx_I$, $d\omega = \sum df_I dx_I$.

d is called the exterior differentiation.

The wedge product of $\omega = \sum f_I dx_I$, $\tau = \sum g_J dx_J$

$\rightsquigarrow$ $\omega \wedge \tau = \sum f_I g_J dx_I dx_J$.

Properties

i) $\omega \wedge \tau = (-1)^{\deg\omega \deg\tau} \tau \wedge \omega$. (graded - commutative)

ii) $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg\omega} \omega \wedge d\tau$ (d : antiderivation)

iii) $d^2 = 0$.

Proof of (iii) for $f \in \Omega^0(\mathbb{R}^n)$,

$$d^2 f = d\left(\sum_j \frac{\partial f}{\partial x_j} dx_j\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j = 0.$$

For $\omega \in \Omega^p(\mathbb{R}^n)$, $\omega = \sum f_I dx_I$

$$d^2\omega = d\left(\sum df_I dx_I\right) = \sum\left(d^2 f_I \wedge dx_I - df_I \wedge d^2 x_I\right)$$
$$= 0.$$

$$\cdots \to \Omega^{p-1}(\mathbb{R}^n) \xrightarrow{d} \Omega^p(\mathbb{R}^n) \xrightarrow{d} \Omega^{p+1}(\mathbb{R}^n) \to \cdots$$

: de Rham complex.

Kernel of d = closed forms.

Image of d = exact forms.

Note: Since $d^2 = 0$, the exact forms are closed

Remark

The de Rham complex can be viewed as a set of differential equation whose solutions are closed forms.
For example, the differential equation $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$ corresponds to the finding a closed 1-form $f dx + g dy$.

Remark On $\mathbb{R}^3$, There is the correspondence

$$
\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathfrak{X}(M) & \xrightarrow{\text{curl}} & \mathfrak{X}(M) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\
① \wr\!\| & & ② \wr\!\| & & ③ \wr\!\| & & ④ \wr\!\| \\
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}
$$

① : $f \leftrightarrow f$.

② : $X = (f_1, f_2, f_3) \longleftrightarrow f_1 dx + f_2 dy + f_3 dz$.

③ : $X = (f_1, f_2, f_3) \longleftrightarrow f_1 dydz - f_2 dxdz + f_3 dxdy$.

④ : $f \longleftrightarrow f\, dxdydz$.

$\int_M d\omega = \int_{\partial M} \omega$

(Stokes' theorem).

$*$ $\left.\begin{array}{l} \text{curl}(\text{grad} f) = 0 \\ \text{div}(\text{curl} F) = 0 \end{array}\right. \longleftrightarrow d^2 = 0.$

Definition The p-th de Rham chomology of $\mathbb{R}^n$ is the quotient vector space defined by

$$H^p_{dR}(\mathbb{R}^n) = \{\text{closed p-forms}\} / \{\text{exact p-forms}\}.$$

* We can define $H^p_{dR}(U)$ for an open set $U \subseteq \mathbb{R}^n$ in the same way.

(i.e. $\Omega^*(U) = C^\infty(U) \otimes_{\mathbb{R}} \Omega^*$.)

<u>Example</u>.

(i) $H^p_{dR}(*) = \begin{cases} \mathbb{R} & p=0 \\ 0 & \text{o.w} \end{cases}$

$$0 \xrightarrow{0} \underset{\mathbb{R}}{\underset{\parallel}{\Omega^0(*)}} \xrightarrow{0} 0$$

(ii) $H^p_{dR}(\mathbb{R}) = \begin{cases} \mathbb{R} & p=0 \\ 0 & \text{o.w} \end{cases}$

$$0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \longrightarrow 0$$

$\Rightarrow$ $\ker d \cap \Omega^0(\mathbb{R}) = \{\text{constant functions on } \mathbb{R}\} = \mathbb{R}$.

$\Rightarrow$ $H^0_{dR}(\mathbb{R}) = \mathbb{R}$.

For $H^1_{dR}(\mathbb{R})$, $\ker d = \Omega^1(\mathbb{R})$.

Let $\omega = f(x)dx$. consider $g(x) = \int_0^x f(t)dt$

$\Rightarrow$ $dg = f(x)dx = \omega$. $\Rightarrow$ $H^1_{dR}(\mathbb{R}) = 0$.

(iii) In general, $H^p_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p=0 \\ 0 & \text{o.w.} \end{cases}$ (Poincaré lemma)

(iv) $H^p_{dR}(\mathbb{R}-\{0\}) = \begin{cases} \mathbb{R}\oplus\mathbb{R} & p=0 \\ 0 & \text{o.w.} \end{cases}$ $\rightsquigarrow$ $\int \frac{1}{x}dx = \begin{cases} \log|x| + C_1 & (x>0) \\ \log|x| + C_2 & (x<0) \end{cases}$

4. Functoriality.

for $\eta \in \Omega^p(N)$,
$(F^*\eta)_p = dF_p^*(\eta_{F(p)})$

Let $F : M \to N$ be a smooth map.

The pullback $F^* : \Omega^*(N) \longrightarrow \Omega^*(M)$ carries closed, exact forms on N into closed, exact forms on M.

And hence, it induces

$$F^* : H^\bullet_{dR}(N) \longrightarrow H^\bullet_{dR}(M).$$

$\rightsquigarrow$ $H_{dR}^\bullet : \mathrm{Man}^\infty \longrightarrow \mathrm{Vect}_{\mathbb{R}}$ defines a contravariant funtor.

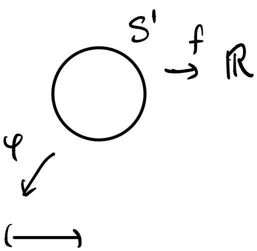
<u>Example</u> $H^\bullet_{dR}(S^1)$

$$0 \xrightarrow{0} \Omega^0(S^1) \xrightarrow{d} \Omega^1(S^1) \xrightarrow{0} 0$$

$$H^0_{dR}(S^1) = \{\text{closed } 0\text{-forms on } S^1\} / \{0\}$$

$$= \{ f : S^1 \to \mathbb{R} \mid df = 0 \}$$

$S^1 \subset \mathbb{R}^2$, consider a coordinate $(\cos\theta, \sin\theta) \overset{\varphi}{\longmapsto} \theta$



$$df = \frac{df}{d\theta}\, d\theta = 0 \iff \frac{df}{d\theta} \equiv 0.$$

$\Rightarrow$ f is constant since S^1 is connected.

$$\Rightarrow \quad H^0_{dR}(S^1) = \mathbb{R}.$$

$$H^1_{dR}(S^1) = \frac{\{\text{closed } 1\text{-forms on } S^1\}}{\{\text{exact } 1\text{-forms on } S^1\}}$$

Note that $\{\text{closed } 1\text{-forms on } S^1\} = \Omega^1(S^1)$.

Let $\xi \in \Omega^1(S^1)$, write $\xi = f(t)\,dt$.

If ξ is exact, i.e. $\xi = df$ for some $f \in \Omega^0(S^1)$

$$\int_{S^1} \xi = \int_{S^1} df = f(2\pi) - f(0) = 0. \qquad \int_{S^1} : \Omega^1(S^1) \to \mathbb{R}$$

linear map.

$\Rightarrow \{\text{exact } 1\text{-forms on } S^1\} \subseteq \ker \int_{S^1}$

If $\eta \in \Omega^1(S^1)$ satisfies $\int_{S^1} \eta = 0$

Write $\eta = f(t)\,dt$, define $g(t) = \int_0^t f(s)\,ds$

$\Rightarrow dg = f(t)\,dt = \eta$.

$\therefore \ker \int_{S^1} = \{\text{exact } 1\text{-forms on } S^1\}$

Hence, Isomorphism theorem says

$$H^1_{dR}(S^1) = \frac{\Omega^1(S^1)}{\ker \int_{S^1}} \cong \operatorname{Im} \int_{S^1} = \mathbb{R}.$$

$\left(\because \int_{S^1} \frac{dt}{2\pi} = 1\right)$