Std Models

Gauge fields $\left\{ \begin{array}{l} \text{Electromagnetism} \\ \text{Weak} \\ \text{Strong} \end{array} \right.$ force

Photon, W, Z described by eqs closely related to Maxwell's eqs.

Gluon

Quatum fields.

힘은 입자로 보기.

Gravity $\longrightarrow$ Einstein / D.G. QT. $\nleftarrow$ hard to quantize.

Recently, ~~me~~ gravity may be closer to the ~~QT~~ Gauge theories of the std. model using 3-dim Knot theory.

Maxwell Eq. $\mathbb{R}^3 \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^3$

$$\left\{ \begin{array}{l} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \partial_t \vec{B} = 0 \end{array} \right. \Rightarrow dF = 0$$

Lorentz force Law.

ex) $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

$$\left\{ \begin{array}{l} \nabla \cdot \vec{E} = \rho \longrightarrow \text{charge density} \\ \nabla \times \vec{B} - \partial_t \vec{E} = \vec{j} \longrightarrow \text{current density} \end{array} \right. \Rightarrow *d*F = J$$

F : electromagnetic field J : current.

Assume Vaccuum.

$\nabla\cdot B = 0$, $\nabla\times E + \partial_t B = 0$.

$\nabla\cdot E = 0$, $\nabla\times B - \partial_t E = 0$

$\Rightarrow$ Transformation. $\vec{B}' \to \vec{E}$, $\vec{E}' \to \vec{B}$. $\Rightarrow$ Symmetry called "duality".

$\Downarrow$

Unified ~~ST~~ that electromagnetic field.

$\Rightarrow \mathcal{E} = E + iB$. duality $\mathcal{E}' \to -i\mathcal{E}$.

$\Rightarrow \nabla\cdot\mathcal{E} = 0$, $\nabla\times\mathcal{E} = i\partial_t\mathcal{E}$.

$$\nabla\times(\nabla\times E) = \nabla\times(-\partial_t B) = -\partial_t(\nabla\times B) = -\partial_t(j + \partial_t E).$$

$$\nabla\times(\nabla\times E) = \underbrace{\nabla(\nabla\cdot E)}_{0} - \Delta E.$$

If $j = 0$ $\Rightarrow \Delta E = \partial_{tt} E$. $\Rightarrow$ Wave eq.

Non-Vacuum.

Consider $\nabla\cdot\mathcal{E} = \rho$, $\nabla\times\mathcal{E} = i(\partial_t\mathcal{E} + j)$.

Let $i\rho_m$; Let ij_m.

$\Rightarrow \nabla\cdot B = \rho_m$, $\nabla\times E + \partial_t B = j_m$. $\rightarrow$ Magnetic monopoles.

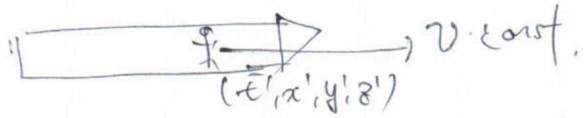
$\nabla\cdot E = \rho_e$, $\nabla\times B - \partial_t E = j_e$.

$\Rightarrow \rho, j \in$ Real.

Einstein. Maxwell $\longleftrightarrow$ Special Relativity

$\Uparrow$

Symmetries of space and time. through Lorentz transformation.



$$t' = (\cosh\phi)t - (\sinh\phi)x$$
$$x' = -(\sinh\phi)t + (\cosh\phi)x$$
$$y' = y$$
$$z' = z.$$

(t, x, y, z)

Spacetime.

ϕ: rapidity s.t. $\tanh\phi = v$.

Un-accelerated $\Rightarrow$ light speed C. (physical)

Have a sol of Maxwell eq (Mathematical)

and do a Lorentz transform of E, B, ρ, j $\Rightarrow$ again have a solution.

$$\rho' = (\cosh\phi)\rho - (\sinh\phi)j_x$$
$$j_x' = -(\sinh\phi)\rho + (\cosh\phi)j_x$$
$$j_y' = j_y$$
$$j_z' = j_z.$$

mixed means that ρ and j are two aspects of a single thing called the current.

$$E_x' = E_x$$
$$E_y' = (\cosh\phi)E_y - (\sinh\phi)B_z$$
$$E_z' = (\sinh\phi)B_y + (\cosh\phi)E_z$$

$$B_x' = B_x$$
$$B_y' = (\cosh\phi)B_y + (\sinh\phi)E_z$$
$$B_z' = -(\sinh\phi)E_y + (\cosh\phi)B_z.$$

∇, $\nabla\cdot$, $\nabla\times$ $\rightarrow d$.

1-form

~~$\nabla f : \mathbb{R}^n \to \mathbb{R}$~~ $\nabla f : T\mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad \nabla f(v) = \nabla f \cdot v = v(f)$.

$\nabla f \cdot v = vf$: keeps track of the directional derivatives of f in all directs.

we want on M. df

∇f : vect fields, $\longleftrightarrow$ df

$\cdot$? $\sim \mathbb{R}^n$? $\quad \Rightarrow df$ × vect fields ×, 1-forms.

Property.

$v \mapsto \nabla f \cdot v$. $\qquad \nabla f \cdot (v+w) = \nabla f \cdot v + \nabla f \cdot w$.

$v \mapsto vf$. $\qquad \nabla f \cdot (gv) = g(\nabla f \cdot v)$.

$\qquad g \in C^\infty(\mathbb{R}^n)$

1-form on M : $\operatorname{Vect}(M) \longrightarrow C^\infty(M)$

linear.

$\omega \cdot (v+w) = \omega(v) + \omega(w)$.

$\omega(gv) = g\,\omega(v)$

Let $f \in C^\infty(M)$.

$df(v) = v(f)$

$df(v+w) = (v+w)f = vf + wf = df(v) + df(w)$

$df(gv) = (gv)(f) = g\,v(f) = g\,df(v)$.

$\Rightarrow df$ is 1-form.

ω, M. ~~[illegible]~~ $\Rightarrow \Omega^1(M)$: $C^\infty(M)$ - Module.

~~[illegible]~~:

$\nabla \cdot B = 0$

$\nabla \times E + \partial_t B = 0$ on M

Grad. $\Omega^0 \to \Omega^1$

Curl. $\Omega^1 \to \Omega^2$.

div. $\Omega^2 \to \Omega^3$.

$$B = B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy.$$

$$E = E_x\, dx + E_y\, dy + E_z\, dz.$$

Space time (t, x^1, x^2, x^3).

$\Rightarrow F = B + E \wedge dt$; Electromagnetic field.

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu.$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

$$dF = d(B + E \wedge dt) = dB + dE \wedge dt.$$

$d_s\omega = \partial_i \omega_I\, dx^i \wedge dI$; Spacelike.

$dt \wedge \partial_t \omega = \partial_0 \omega_I\, dx^0 \wedge dx^I$: Timelike.

$$d_s B + dt \wedge \partial_t B + (d_s E + dt \wedge \partial_t E) \wedge dt.$$

$$= \underbrace{d_s B}_{=0} + \underbrace{(\partial_t B + d_s E)}_{=0} \wedge dt.$$

Semi-Riemannian metric. $g : V \times V \to \mathbb{R}$

· bilinear, symmetric, nondegenerate.

if $g(v,w) = 0$ for all $w \in V$ $\Rightarrow v = 0$.

$g_p : T_pM \times T_pM \to \mathbb{R}$.

Signature of g $(n, 0)$: Riemannian.

$(n-1, 1)$. Lorentzian.

Lorentzian metric $g = -dt^2 + {}^3g$

$\underline{\quad}$ space Riemannian.

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \; 0 \; 0 \\ 0 & \\ 0 & {}^3g_{ij} \\ 0 & \end{pmatrix}$$

$g_{\mu\nu} = g(e_\mu, e_\nu)$.

$g^{\mu\nu}$ = inverse.

Inner product of ω & μ. $\langle \omega, \mu \rangle$

$g(v,w) = g_{\alpha\beta} v^\alpha w^\beta$. $\Rightarrow \langle \omega, \mu \rangle = g^{\alpha\beta} \omega_\alpha \mu_\beta$.

p-form $\langle e^1 \wedge \cdots \wedge e^p, f^1 \wedge \cdots \wedge f^p \rangle = \det [g(e^i, f^j)]$.

$$(*F)_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

$E_i \mapsto -B_i$
$B_i \mapsto E_i$

$j = j_1 dx^1 + j_2 dx^2 + j_3 dx^3$

$\underline{J}$ = $\cancel{\rho\, dx_0 + j}$
current $\rho\,\partial_0 + j^1\partial_1 + j^2\partial_2 + j^3\partial_3$

Using Minkowski metric
$J = j - \rho\, dt$

$$\left.\begin{aligned} \nabla\cdot E &= \rho \\ \nabla\times B - \partial_t E &= \vec{j} \end{aligned}\right\} \Rightarrow \; *d*F = J$$

$\to$ $*_S d_S *_S E = \rho$
$-\partial_t E + *_S d_S *_S B = j$

$*F = *_S E - *_S B \wedge dt$
$d*F = *_S \partial_t E \wedge dt + d_S *_S E - d_S *_S B \wedge dt$
$*d*F = -\partial_t E - *_S d_S *_S E \wedge dt + *_S d_S *_S B$
$= j - \rho\, dt$ $\Rightarrow$ $*_S d_S *_S E = \rho$
$-\partial_t E + *_S d_S *_S B = j$