The Gromov-Hausdorff distance between spheres

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April 11th, 2024

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- Despite being widely used in Riemannian geometry, precise value of the Gromov-Hausdorff distance between spaces is largely unknown.
- This is because geometers primarily focus on the topology induced by d_{GH} (such as Gromov's precompactness theorem and the finiteness of homotopy types) as well as the convergence with respect to d_{GH} (for instance, the Gromov-Hausdorff limit of Riemannian manifolds being Alexandrov spaces).

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- In particular, the **stability theorem** in Topological Data Analysis (TDA) was important motivation for this project. So, let's look at this theorem more carefully.

Overview

Definition (Vietoris-Rips filtration)

Let (X, d_X) be a metric space and $r > 0$. The (open) **Vietoris-Rips complex** $\mathrm{VR}(X; r)$ of X is the simplicial complex such that

$$\mathrm{VR}(X; r) := \{\{x_0, \ldots, x_n\} \subseteq X : \mathrm{diam}(\{x_0, \ldots, x_n\}) < r \text{ for any } n \geq 0\}.$$

Note that if $r \leq s$, then $\mathrm{VR}(X; r)$ is contained in $\mathrm{VR}(X; s)$.

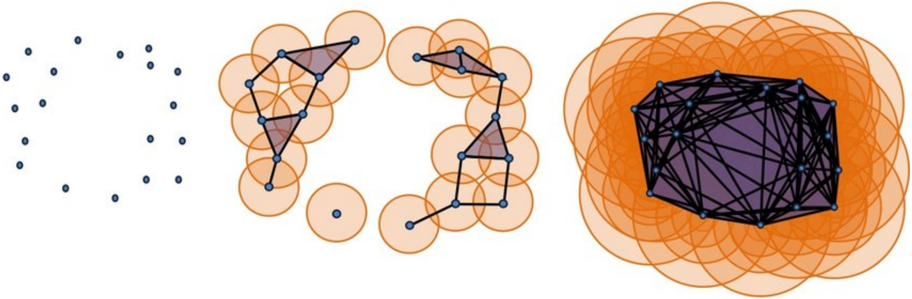
Hence, the family

$$(\mathrm{VR}(X; r), i_{r,s})_{0 < r \leq s}$$

is called the **Vietoris-Rips filtration** of X.

Overview

Figure: An example of Vietoris-Rips filtration (from [Piangerelli et al., 2018])



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- If we apply the k-dimensional homology functor $\mathrm{H}_k(\,\cdot\,;\mathbb{F})$ to $(\mathrm{VR}(X;r), i_{r,s})_{0<r\leq s}$, we get the filtration of $\mathbb{F}$-vector spaces

$$(\mathrm{H}_k(\mathrm{VR}(X;r);\mathbb{F}), (i_{r,s})_*)_{0<r\leq s}.$$

It is denoted by $\mathrm{PH}_k(\mathrm{VR}(X;*);\mathbb{F})$ and called **Vietoris-Rips persistent homology**.

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- If a metric space (X, d_X) is "nice" (for exmaple, totally bounded), then there is a unique multiset of intervals $\{I_\lambda\}_\lambda$ associated to (in a certain rigorous sense) to $\mathrm{PH}_k(\mathrm{VR}(X;*);\mathbb{F})$. Each interval represents "lifespan" of each k-dimensional hole of VR filtration.

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- This multiset is said to be **persistence barcode** and denoted by $\mathrm{barc}_k^{\mathrm{VR}}(X;\mathbb{F})$.

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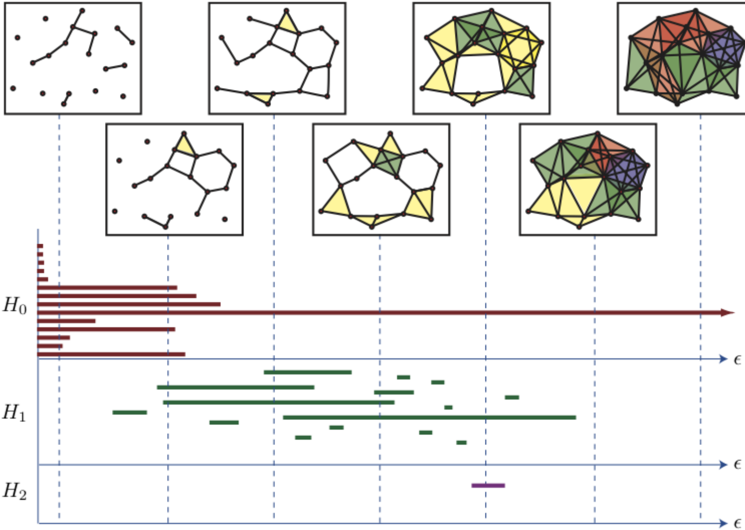
Figure: An Example of persistence barcode (from `datawarrior.wordpress.com`)



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- Moreover, for any two compact metric spaces (X, d_X),(Y, d_Y) and any field $\mathbb{F}$, we have

$$\frac{1}{2}\sup_{k\geq 0} d_{\mathrm{B}}(\mathrm{barc}_k^{\mathrm{VR}}(X;\mathbb{F}), \mathrm{barc}_k^{\mathrm{VR}}(Y;\mathbb{F})) \leq d_{\mathrm{GH}}(X, Y).$$

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  So, persistence barcodes are **stable** w.r.t metric perturbation.
- How good is LHS as an estimator of $d_{\mathrm{GH}}(X, Y)$?

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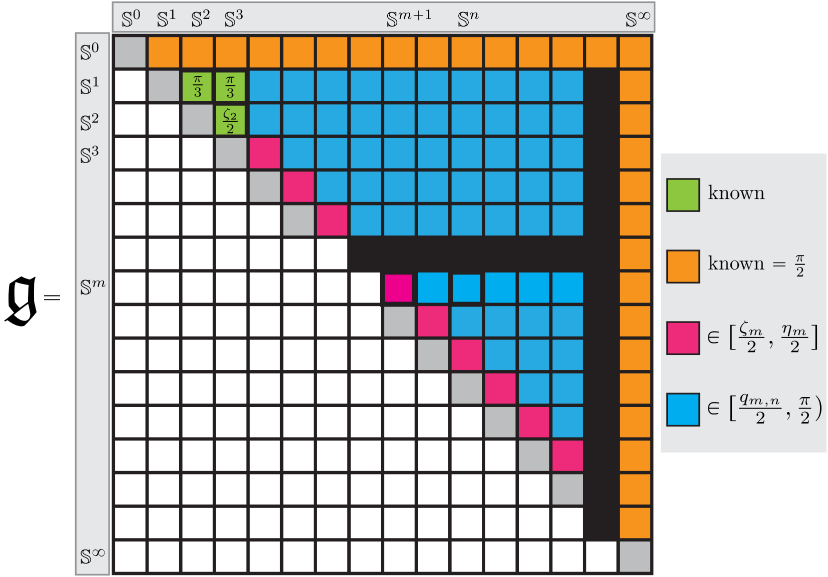
  for $0 \leq m < n \leq \infty$.
- Why is it important? the n-sphere serves as a model space with a single n-dimensional hole, making it a natural foundation for comprehending Gromov-Hausdorff distances before delving into more intricate spaces. Also, the theoretical result will be a reference for the other computations.

The matrix of $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$



Preliminaries

- Given two sets X and Y, a correspondence between them is any relation $R \subseteq X \times Y$ such that $\pi_X(R) = X$ and $\pi_Y(R) = Y$. $\mathcal{R}(X, Y)$ denotes the set of all correspondences between X and Y.

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- For any relation $R \subseteq X \times Y$, the distortion of R is defined in the following way: $\mathrm{dis}(R) := \sup_{(x,y),(x',y') \in R} \left| d_X(x, x') - d_Y(y, y') \right|$.

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$$d_{\mathrm{GH}}(X, Y) := \frac{1}{2} \inf_{R\in\mathcal{R}(X,Y)} \operatorname{dis}(R),$$

- By the above characterization, it is easy to check $d_{\mathrm{GH}}(X, Y) \leq \frac{\max\{\mathrm{diam}(X),\mathrm{diam}(Y)\}}{2}$. Hence, $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \leq \frac{\pi}{2}$ for all $0 \leq m < n \leq \infty$.

Gromov-Hausdorff distance and topology of spheres

- The topology of spheres play an important role in order to build bounds of $d_{\mathrm{GH}}(\mathbb{S}^m,\mathbb{S}^n)$. In particular, for lower bounds.

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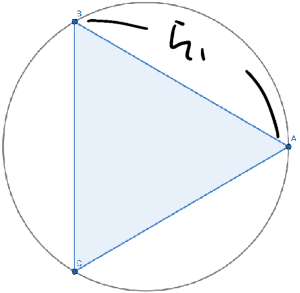
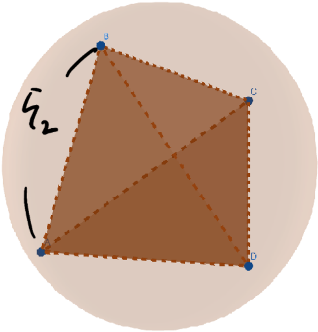
- The topology of spheres play an important role in order to build bounds of $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$. In particular, for lower bounds.
- For an example, if we use persistent homology and stability of the bottleneck distance [Lim et al., 2020]:

$$d_{\mathrm{GH}}(\mathbb{S}^{m}, \mathbb{S}^{n}) \geq \frac{1}{2} \sup_k d_{\mathrm{B}}(\mathrm{barc}_k^{\mathrm{VR}}(\mathbb{S}^{m}; \mathbb{F}), \mathrm{barc}_k^{\mathrm{VR}}(\mathbb{S}^{n}; \mathbb{F})) = \frac{1}{4}\zeta_{m}$$

where $\zeta_m := \arccos\left(-\frac{1}{m+1}\right)$ for any $0 < m < n < \infty$.





$\zeta_m := \arccos\left(-\frac{1}{m+1}\right)$ *is the geodesic distance between any two vertices of a regular* $(m+1)$*-simplex inscribed in* $\mathbb{S}^m$*.*

Example

Note that $\zeta_1 = \frac{2\pi}{3}$, and $\zeta_2 \approx 1.91$.

Gromov-Hausdorff distance and topology of spheres

- Topology plays an important role to build bounds of $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$. In particular, for lower bounds.

- For an example, if we use persistent homology and stability of the bottleneck distance:

$$\begin{aligned} d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) &\geq \frac{1}{2} d_{\mathrm{B}}(\mathrm{barc}_m^{\mathrm{VR}}(\mathbb{S}^m; \mathbb{F}), \mathrm{barc}_m^{\mathrm{VR}}(\mathbb{S}^n; \mathbb{F})) \\ &\geq \frac{1}{2} \mathrm{FillRad}(\mathbb{S}^m) = \frac{1}{4} \zeta_m := \frac{1}{4} \arccos\left(-\frac{1}{m+1}\right) \end{aligned}$$

  for any $0 < m < n < \infty$.

- However, Borsuk-Ulam type theorems give better results!

$d_{\mathrm{GH}}(\mathbb{S}^0, \mathbb{S}^n) = d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^\infty) = \frac{\pi}{2}$

Theorem (Lyusternik-Schnirelmann)

Let $n \in \mathbb{N}$, and $\{U_1, \ldots, U_{n+1}\}$ be a closed cover of $\mathbb{S}^n$. Then there is $i_0 \in \{1, \ldots, n+1\}$ such that U_{i_0} contains two antipodal points.

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Theorem ([Lim et al., 2021, Proposition 1.5,1.6])

For any $n > 0$ and $m < \infty$, $d_{\mathrm{GH}}(\mathbb{S}^0, \mathbb{S}^n) = d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^\infty) = \frac{\pi}{2}$.

Proof.

Fix arbitrary correspondence $R \in \mathcal{R}(\mathbb{S}^0, \mathbb{S}^n)$. Recall that $\mathbb{S}^0 = \{-1, 1\}$. Let $A^- := \{x \in \mathbb{S}^n : (-1, x) \in R\}$, and $A^+ := \{x \in \mathbb{S}^n : (1, x) \in R\}$. Then, by Lyusternik-Schnirelmann Theorem, A^- or A^+ contains a pair of antipodal points. It implies $d_{\mathrm{GH}}(\mathbb{S}^0, \mathbb{S}^n) \geq \frac{\pi}{2}$ so that $d_{\mathrm{GH}}(\mathbb{S}^0, \mathbb{S}^n) = \frac{\pi}{2}$. The proof of $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^\infty) = \frac{\pi}{2}$ is similar, but require a bit more details so I omit it. $\square$

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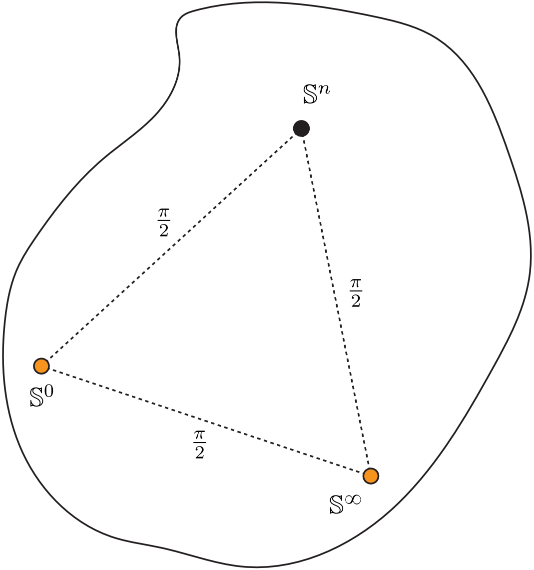

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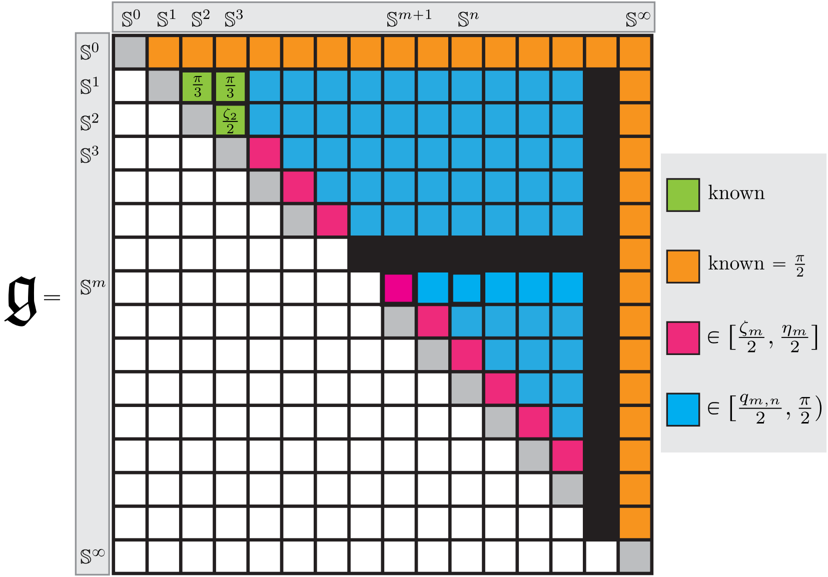

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- Then, one might conjecture $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) = \frac{\pi}{2}$ for any $0 \leq m < n \leq \infty$. But, it is NOT TRUE!

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- Actually, we were able to prove that $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) < \frac{\pi}{2}$ for any $0 < m < n < \infty$. In order to prove this result, we established the following **Reverse Borsuk-Ulam Theorem**.

Theorem (Reverse Borsuk-Ulam Theorem [Lim et al., 2021])

For all integers $0 < m < n < \infty$, there exists an odd (ie, $\psi_{m,n}(-x) = -\psi_{m,n}(x)$) continuous surjection

$$\psi_{m,n} : \mathbb{S}^m \twoheadrightarrow \mathbb{S}^n.$$

Theorem ([Lim et al., 2021, Theorem A])

$d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) < \frac{\pi}{2}$ for all $0 < m < n < \infty$.

Proof.

Let $\psi_{m,n} : \mathbb{S}^m \twoheadrightarrow \mathbb{S}^n$ be the map given in Reverse Borsuk-Ulam Theorem. Since $\psi_{m,n}$ is surjective, the graph of $\psi_{m,n}$ is a correspondence between $\mathbb{S}^m$ and $\mathbb{S}^n$. Hence, it is enough to show $\mathrm{dis}(\psi_{m,n}) < \pi$. Since $\mathbb{S}^m, \mathbb{S}^n$ are compact, there exists $x_0, x_0' \in \mathbb{S}^m$ attaining the $\mathrm{dis}(\psi_{m,n})$:

$$\mathrm{dis}(\psi_{m,n}) = |d_{\mathbb{S}^m}(x_0, x_0') - d_{\mathbb{S}^n}(\psi_{m,n}(x_0), \psi_{m,n}(x_0'))|.$$

if $x_0' = -x_0$, then $\mathrm{dis}(\psi_{m,n}) = 0$ since $\psi_{m,n}$ is odd. This implies that $\psi_{m,n}$ is surjective isometry which is contradiction. Hence, one can conclude that $0 < d_{\mathbb{S}^m}(x_0, x_0') < \pi$ therefore $\mathrm{dis}(\psi_{m,n}) < \pi$ as we required. $\square$

Proof of Reverse Borsuk-Ulam Theorem

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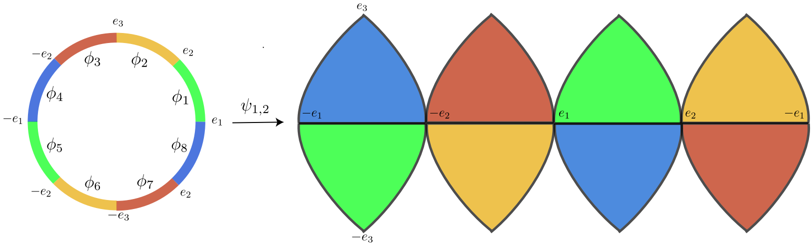
- By using the space of filling curve, one can construct a continuous surjective map from each segment to each triangle. By combining eight of them, we establish the required $\psi_{1,2}$ as follows:



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- By using the suspension, for any map $\psi : \mathbb{S}^m \to \mathbb{S}^n$, one can build a map $S\psi : S\mathbb{S}^m = \mathbb{S}^{m+1} \to S\mathbb{S}^n = \mathbb{S}^{n+1}$.

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- So, by applying the suspension inductively, one can get continuous, surjective, and odd map $\psi_{m,m+1} : \mathbb{S}^m \twoheadrightarrow \mathbb{S}^{m+1}$.
- Finally, for arbitrary $0 < m < n < \infty$,

$$\psi_{m,n} := \psi_{n-1,n} \circ \psi_{n-2,n-1} \circ \cdots \circ \psi_{m,m+1}.$$

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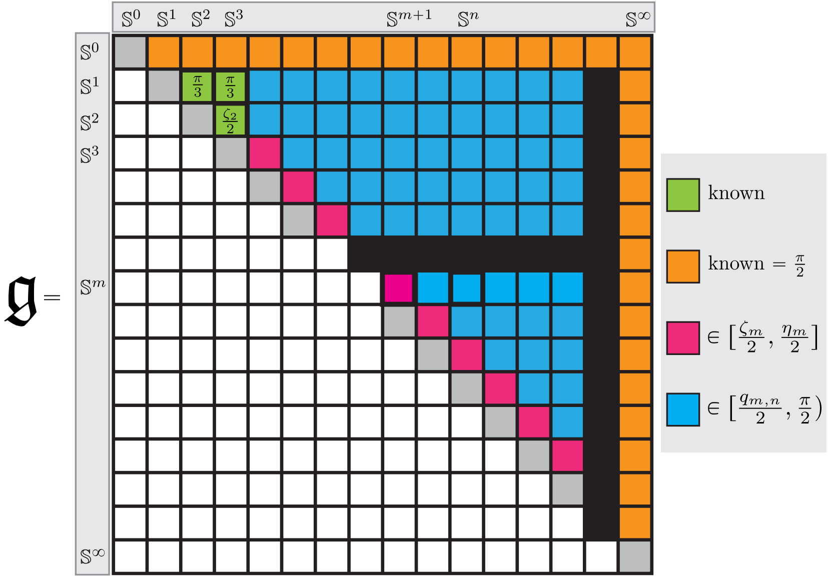

Quantitative Borsuk-Ulam and $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2}\zeta_m$

- We were able to prove

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- Moreover, we were able to prove this bound is tight when $(m, n) = (1, 2), (1, 3), (2, 3)$.

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- Moreover, we were able to prove this bound is tight when $(m, n) = (1, 2), (1, 3), (2, 3)$.
- Because of the following alternative characterization of the Gromov-Hausdorff distance:

$$d_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_{\phi: X \to Y, \psi: Y \to X} \max\{\mathrm{dis}(\phi), \mathrm{dis}(\psi), \mathrm{codis}(\phi, \psi)\},$$

  we have $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2} \inf_{\psi: \mathbb{S}^n \to \mathbb{S}^m} \mathrm{dis}(\psi)$ over all (not-necessarily continuous) maps $\psi : \mathbb{S}^n \to \mathbb{S}^m$.

Quantitative Borsuk-Ulam and $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2}\zeta_m$

- Lyusternik-Schnirelmann Theorem is equivlanet to the classical Borsuk-Ulam Theorem:

  "For any continuous map $g : \mathbb{S}^{m+1} \to \mathbb{S}^m$, g cannot be odd."

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- The modulus of discontinuity of a map $g : X \longrightarrow Y$,

  $$\inf\{\delta \geq 0 : \forall x \in X, \exists \text{ open } U_x \ni x \text{ s.t. } \mathrm{diam}(g(U_x)) \leq \delta\}$$

  is a quantity measuring the discontinuity of g.

Quantitative Borsuk-Ulam and $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2}\zeta_m$

- Lyusternik-Schnirelmann Theorem is equivlanet to the classical Borsuk-Ulam Theorem:

  "For any continuous map $g : \mathbb{S}^{m+1} \to \mathbb{S}^m$, g cannot be odd."

- But, since Gromov-Hausdorff distance does not require continuity, it is reasonable guess that we may need "discontinuous" version of Borsuk-Ulam.

  "If $g : \mathbb{S}^{m+1} \to \mathbb{S}^m$ is odd, quantitatively speaking, how much discontinuous is g?"

- The modulus of discontinuity of a map $g : X \longrightarrow Y$,

  $$\inf\{\delta \geq 0 : \forall x \in X, \exists \text{ open } U_x \ni x \text{ s.t. } \mathrm{diam}(g(U_x)) \leq \delta\}$$

  is a quantity measuring the discontinuity of g.

- We realized that the modulus of discontinuity of a map g is upper bounded by $\mathrm{dis}(g)$.

Quantitative Borsuk-Ulam and $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2}\zeta_m$

Theorem ([Dubins and Schwarz, 1981, Corollary 3])

For each $m > 0$, if a map $g : \mathbb{S}^{m+1} \to \mathbb{S}^m$ is odd, then the modulus of discontinuity of g is greater than or equal to ζ_m.

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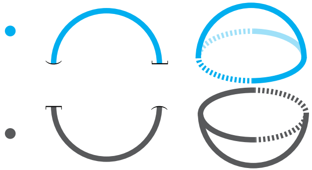


- Let

$$A(\mathbb{S}^0) := \text{one of the two points of } \mathbb{S}^0, \text{ and}$$
$$A(\mathbb{S}^{m+1}) := (\text{strict upper hemisphere of } \mathbb{S}^{m+1}) \cup A(\mathbb{S}^m)$$

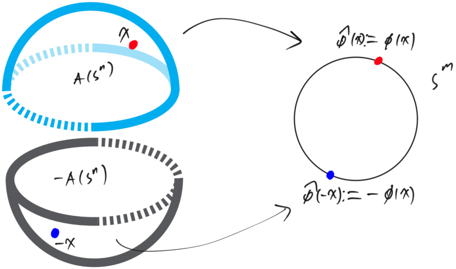
for each $m \geq 0$. Here, we are identifying the equator of $\mathbb{S}^{m+1}$ and $\mathbb{S}^m$.

- Observe that $A(\mathbb{S}^m) \cap (-A(\mathbb{S}^m)) = \emptyset$ and $A(\mathbb{S}^m) \cup (-A(\mathbb{S}^m)) = \mathbb{S}^m$ for each $m \geq 0$ (ie, $A(\mathbb{S}^m)$ is a $\mathbb{Z}_2$-fundamental domain).

Lemma (distortion preserving lemma)

For any $\phi : A(\mathbb{S}^n) \longrightarrow \mathbb{S}^m$, *we have* $\mathrm{dis}(\phi) = \mathrm{dis}(\widehat{\phi})$ *where*

$$\widehat{\phi} : \mathbb{S}^n \longrightarrow \mathbb{S}^m \text{ s.t. } \begin{cases} x & \longmapsto \phi(x) \\ -x & \longmapsto -\phi(x) \end{cases} \forall x \in A(\mathbb{S}^n).$$



Theorem ([Lim et al., 2021, Theorem B])

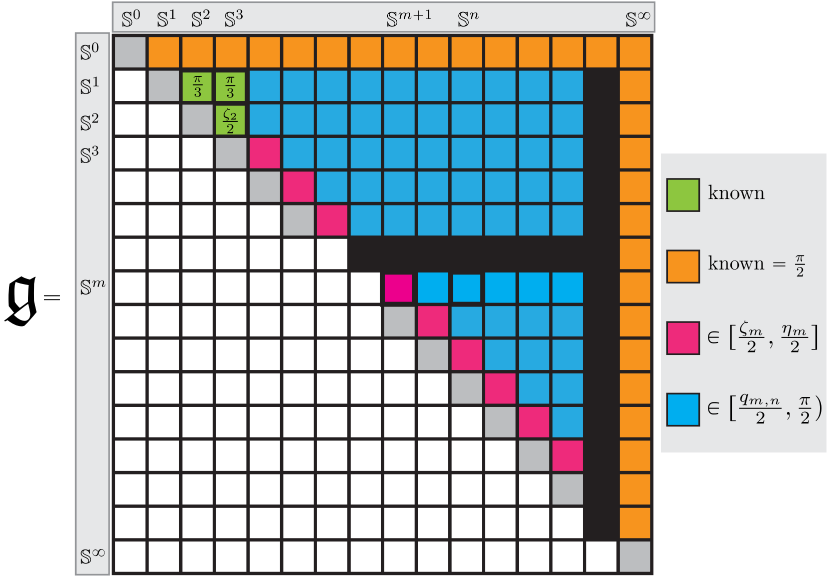
For any $0 < m < n < \infty$, $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2}\zeta_m$.

sketch of proof.

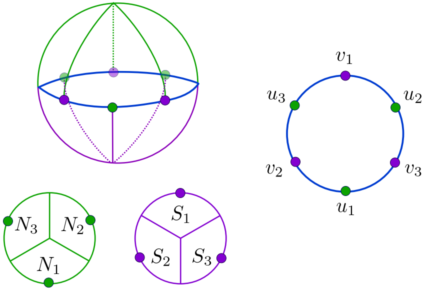
Suppose not. Then, $\exists R \in \mathcal{R}(\mathbb{S}^m, \mathbb{S}^n)$ such that $\mathrm{dis}(R) < \zeta_m$. Since $n > m$, $A(\mathbb{S}^{m+1}) \subset \mathbb{S}^{m+1} \subseteq \mathbb{S}^n$. Hence, one can construct a map $\phi : A(\mathbb{S}^{m+1}) \to \mathbb{S}^m$ such that $(x, \phi(x)) \in R$ for any $x \in A(\mathbb{S}^{m+1})$. Note that $\mathrm{dis}(\phi) \leq \mathrm{dis}(R)$.

Now, apply the distortion preserving lemma, we get an odd map $\widehat{\phi} : \mathbb{S}^{m+1} \to \mathbb{S}^m$ such that $\mathrm{dis}(\widehat{\phi}) = \mathrm{dis}(\phi) \leq \mathrm{dis}(R) < \zeta_m$, which contradicts the Dubins-Schwarz's result. □

Quantitative Borsuk-Ulam and $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2}\zeta_m$



$d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^2) \leq \frac{\pi}{3}$, hence $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^2) = \frac{\pi}{3}$



$$\phi_{2,1} : \mathbb{S}^2 \twoheadrightarrow \mathbb{S}^1 \text{ s.t. } x \longmapsto \begin{cases} u_i & \text{if } x \in N_i \\ x & \text{if } x \in \mathbb{S}^1 \\ v_i = -u_i & \text{if } x \in N_i \end{cases}$$

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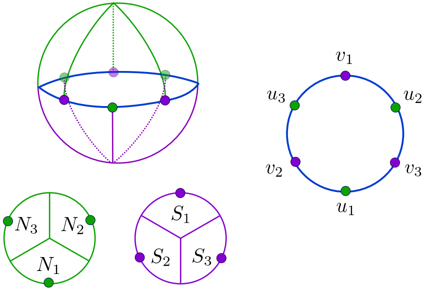

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- $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^2) \leq \frac{\pi}{3}$ since $\mathrm{dis}(\phi_{2,1}) = \frac{2\pi}{3}$. So, $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^2) = \frac{\pi}{3}$.

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$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r, dotted, two heads, "\phi_{3,1}"] \arrow[d, "h"'] & \mathbb{S}^1 \\
\mathbb{S}^2\times[0,\pi) \arrow[r, two heads, "\phi_{2,1}\times\mathrm{id}"] & \mathbb{S}^1\times[0,\pi) \arrow[u, two heads, "T_\bullet"']
\end{tikzcd}$$

- For each $q\in\mathbb{S}^3$, there is a way to choose the unique pair, the point $p_q,\in\mathbb{S}^2$ and the angle $\alpha_q\in[0,\pi)$ such that $q=T_{\alpha_q}p_q$ (rotation of p_q by angle α_q). Let $h(q):=(p_q,\alpha_q)$.

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- Let $\phi_{3,1}(q) := T_{\alpha_q}\phi_{2,1}(p_q)$. One can prove $\mathrm{dis}(\phi_{3,1}) = \frac{2\pi}{3}$.

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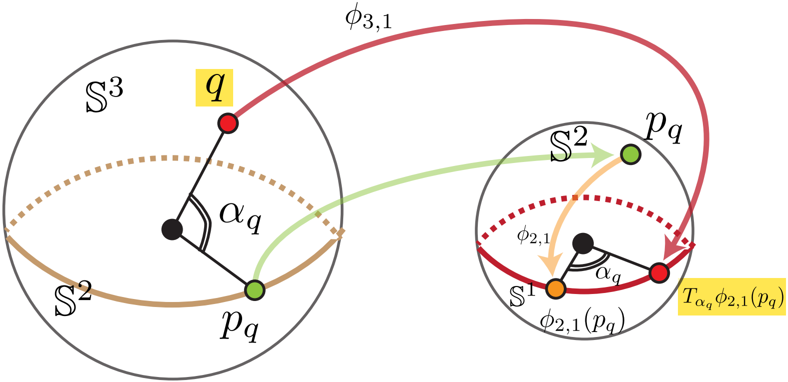

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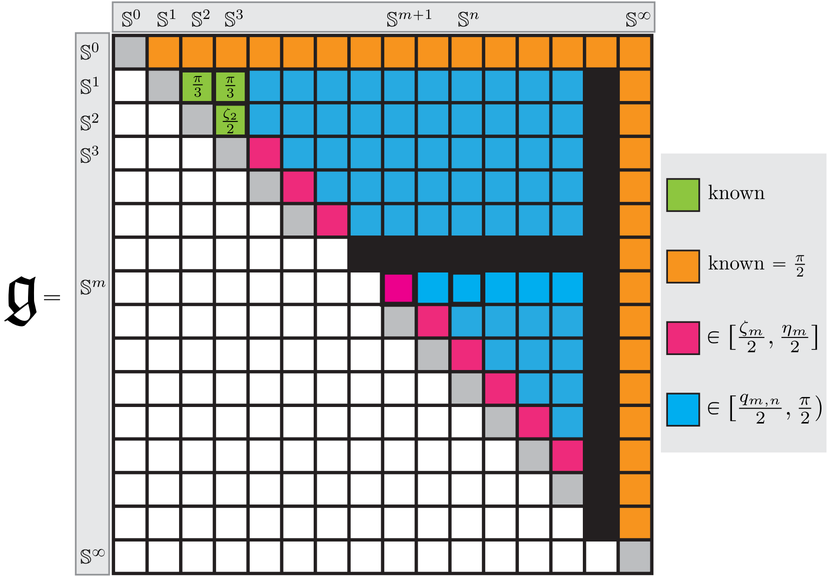
- Let $\phi_{3,1}(q) := T_{\alpha_q}\phi_{2,1}(p_q)$. One can prove $\mathrm{dis}(\phi_{3,1}) = \frac{2\pi}{3}$.
- If we consider $\pi_1 \circ h : \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2$, $(\pi_1 \circ h)^{-1}(\{p, -p\})$ is isometric to $\mathbb{S}^1$ for any $p \in \mathbb{S}^2 \backslash \mathbb{S}^1$. So there is a certain degree of similarity to "Hopf fibration".

The matrix of $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$



Open Questions

- Prove/Disprove the conjecture $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^{m+1}) = \frac{1}{2}\zeta_m$ for $m \geq 3$.
- For fixed m, is $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$ non-decreasing with respect to n? If so, how fast?
- What if we use **Euclidean** distance instead of geodesic distance? Surprisingly, they don't seem directly come from the geodesic cases.
- There is an analogue of Gromov-Hausdorff distance for metric measure spaces, so called "Gromov-Wasserstein distances". Can we compute the Gromov-Wasserstein distances between spheres?

Stronger quantitative Borsuk-Ulam Theorem

- Recall that we proved $2 \cdot d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \zeta_m$ for any $0 < m < n < \infty$. But this lower bound ζ_m only depends on the dimension of the lower dimensional sphere. How can we upgrade this lower bound in a way that it depends on both m and n?

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- This number $\zeta_m := \arccos\left(\frac{-1}{m+1}\right)$ also appears in the following problem:
  "*When the homotopy type of the Vietoris-Rips complex $\mathrm{VR}(\mathbb{S}^m; r)$ changes?*"
  In [Lim et al., 2020], we proved that ζ_m **is the first critical point** of the above problem. More precisely, we know that $\mathrm{VR}(\mathbb{S}^m; r) \simeq \mathbb{S}^m$ for all $0 < r \leq \zeta_m$ and the homotopy type of $\mathrm{VR}(\mathbb{S}^m; r)$ must change for $r > \zeta_m$.

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- Is this coincidence? It turns out that it is not!

Stronger quantitative Borsuk-Ulam Theorem

Theorem (Stronger Quantitatve Borsuk-Ulam Theorem [Adams et al., 2022, Main Theorem])

For all $0 < m < n < \infty$, *if a map* $g : \mathbb{S}^n \to \mathbb{S}^m$ *is odd, then the distortion* $\mathrm{dis}(g)$ *is greater than or equal to* $c_{m,n}$ *where*

$$c_{m,n} := \inf\{r > 0 : \exists \text{ an odd continuous map } \mathbb{S}^n \to |\mathrm{VR}(\mathbb{S}^m; r)|\}.$$

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Theorem ([Adams et al., 2022, Main Theorem])

For any $0 < m < n < \infty$*,* $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n) \geq \frac{1}{2} c_{m,n}$.

Known values of $c_{m,n}$

- One can prove that $c_{m,n} \leq c_{m',n'}$ if $m \geq m'$ and $n \leq n'$.

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- For all $k \geq 1$, $c_{1,2k} = c_{1,2k+1} = \frac{2\pi k}{2k+1}$. Hence, we have $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^{2k}), d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^{2k+1}) \geq \frac{\pi k}{2k+1}$.

Proof of Stronger quantitative Borsuk-Ulam Theorem

Lemma

A map $f : X \to Y$ between metric spaces induces a simplicial map $\overline{f} : \mathrm{VR}(X; r) \to \mathrm{VR}(Y; r + \mathrm{dis}(f))$ for any $r > 0$. Moreover, if f is an odd map, then $\overline{f}$ is also odd.

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Lemma

An arbitrary $\varepsilon > 0$ is given. Suppose a finite subset $X \subset \mathbb{S}^n$ is $\frac{\varepsilon}{2}$-net and $\mathbb{Z}_2$-invariant (ie, $X = -X$), then there exists a continuous odd map $\phi : \mathbb{S}^n \to \mathrm{VR}(X; \varepsilon)$.

Proof of Stronger quantitative Borsuk-Ulam Theorem

Proof.

Fix an arbitrary $\varepsilon > 0$ and choose a finite $\mathbb{Z}_2$-invariant $\frac{\varepsilon}{2}$-net $X \subset \mathbb{S}^n$. Then, by the second lemma in the previous page, there is a continuous odd map $\phi : \mathbb{S}^n \to \mathrm{VR}(X;\varepsilon)$.

Also, by the first lemma in the previous page, the odd map $g|_X$ induces a continuous odd map from $\mathrm{VR}(X;\varepsilon)$ to $\mathrm{VR}(\mathbb{S}^m;\mathrm{dis}(g)+\varepsilon)$.

Then, their composition is a continuous odd map from $\mathbb{S}^n$ to $\mathrm{VR}(\mathbb{S}^m;\mathrm{dis}(g)+\varepsilon)$. By the definition of $c_{m,n}$, we know that $\mathrm{dis}(g)+\varepsilon \geq c_{m,n}$. Since the choice of $\varepsilon > 0$ is arbitrary, finally one can conclude that

$$\mathrm{dis}(g) \geq c_{m,n}.$$

□

The End

Other bounds of $d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$

- $q_{m,n} = \max\left(\frac{\zeta_m}{2}, \frac{\pi}{2} - \mathrm{cov}_{\mathbb{S}^m}(n+1)\right)$ where $\mathrm{cov}_X(k)$ denotes the k-th **covering radius** of X:

$$\mathrm{cov}_X(k) := \inf\{d_{\mathrm{H}}(X,P) \,|\, P \subset X \text{ s.t. } |P| \leq k\}.$$

- η_m is the diameter of the spherical convex hull induced by $\{u_1, \cdots, u_{m+1}\}$ where $\{u_1, \cdots, u_{m+1}, u_{m+2}\}$ are the vertices of a regular $(m+1)$-simplex inscribed in $\mathbb{S}^m$. Hence,

$$\eta_m = \begin{cases} \arccos\left(-\frac{m+1}{m+3}\right) & \text{for } m \text{ odd} \\ \arccos\left(-\sqrt{\frac{m}{m+4}}\right) & \text{for } m \text{ even.} \end{cases}$$

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