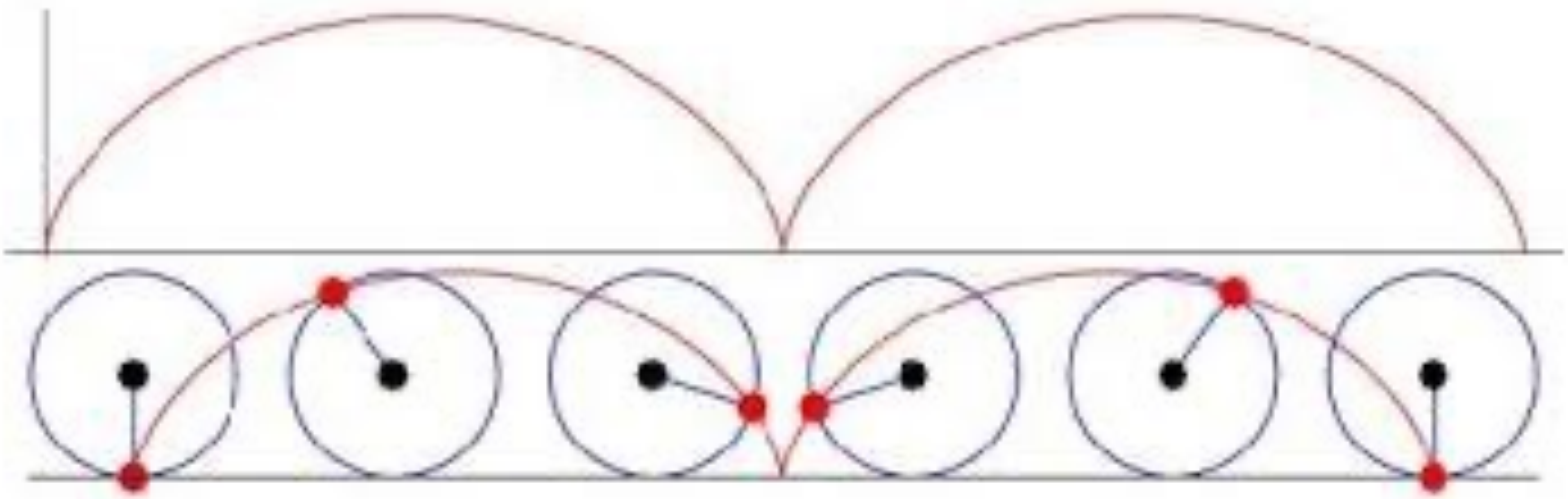
The Euler-Lagrange Equation

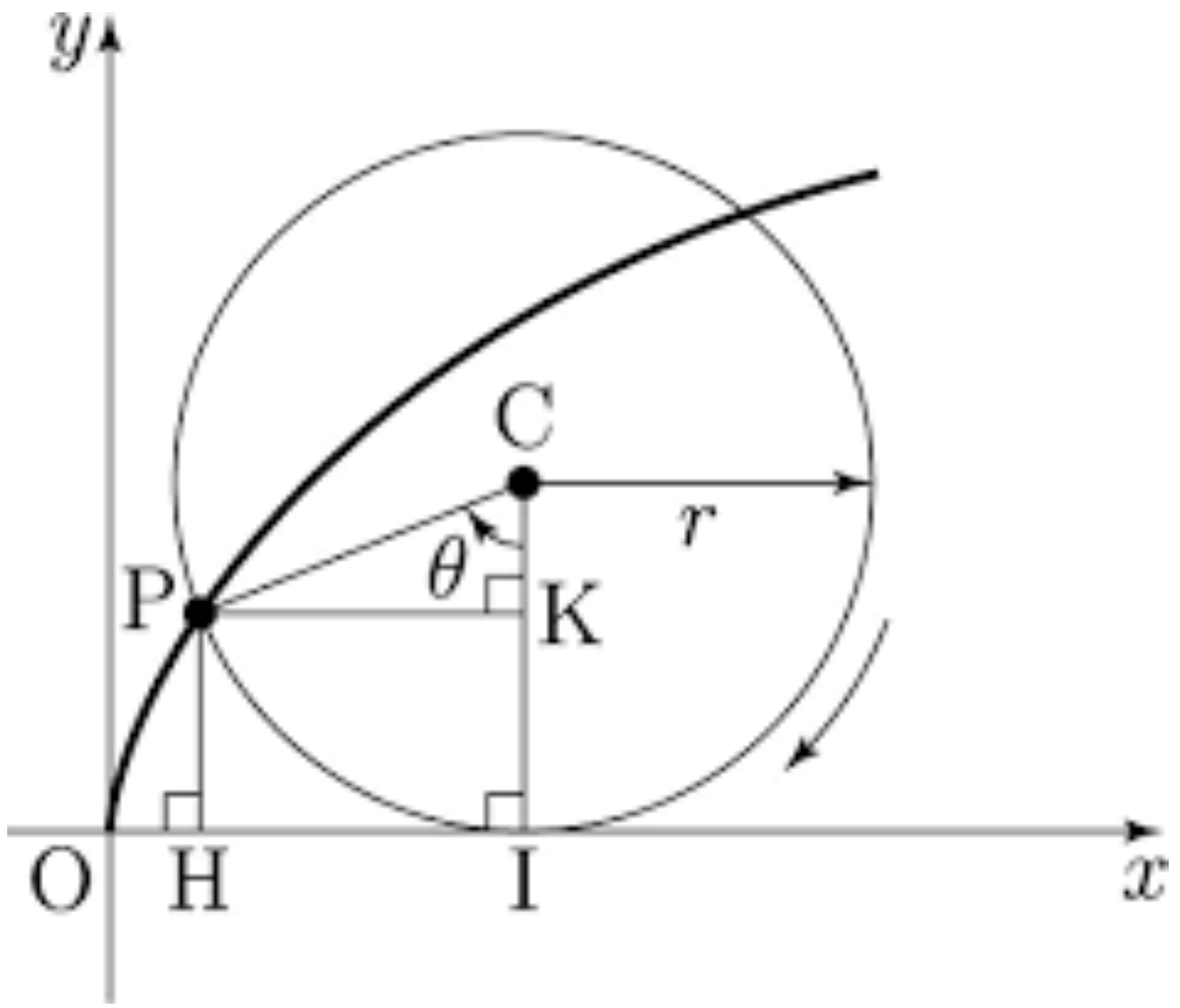
#cycloid #brachistochrone #calculus of variations

3.19.25 주현욱

Contents

A cycloid



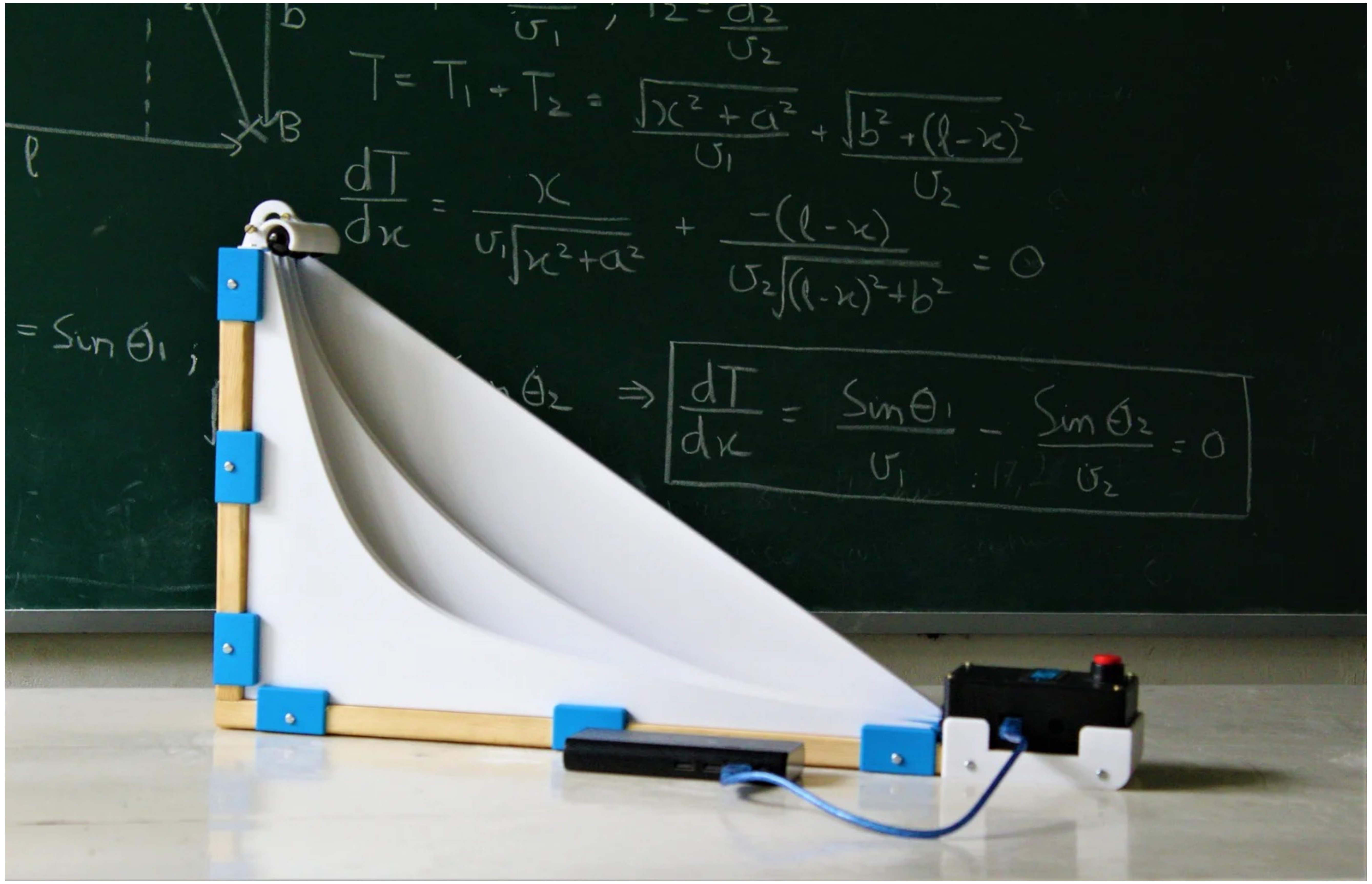


$$x = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = r - r\cos\theta = r(1 - \cos\theta)$$

Brachistochrone

The fastest route between two points

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$



Brachistochrone

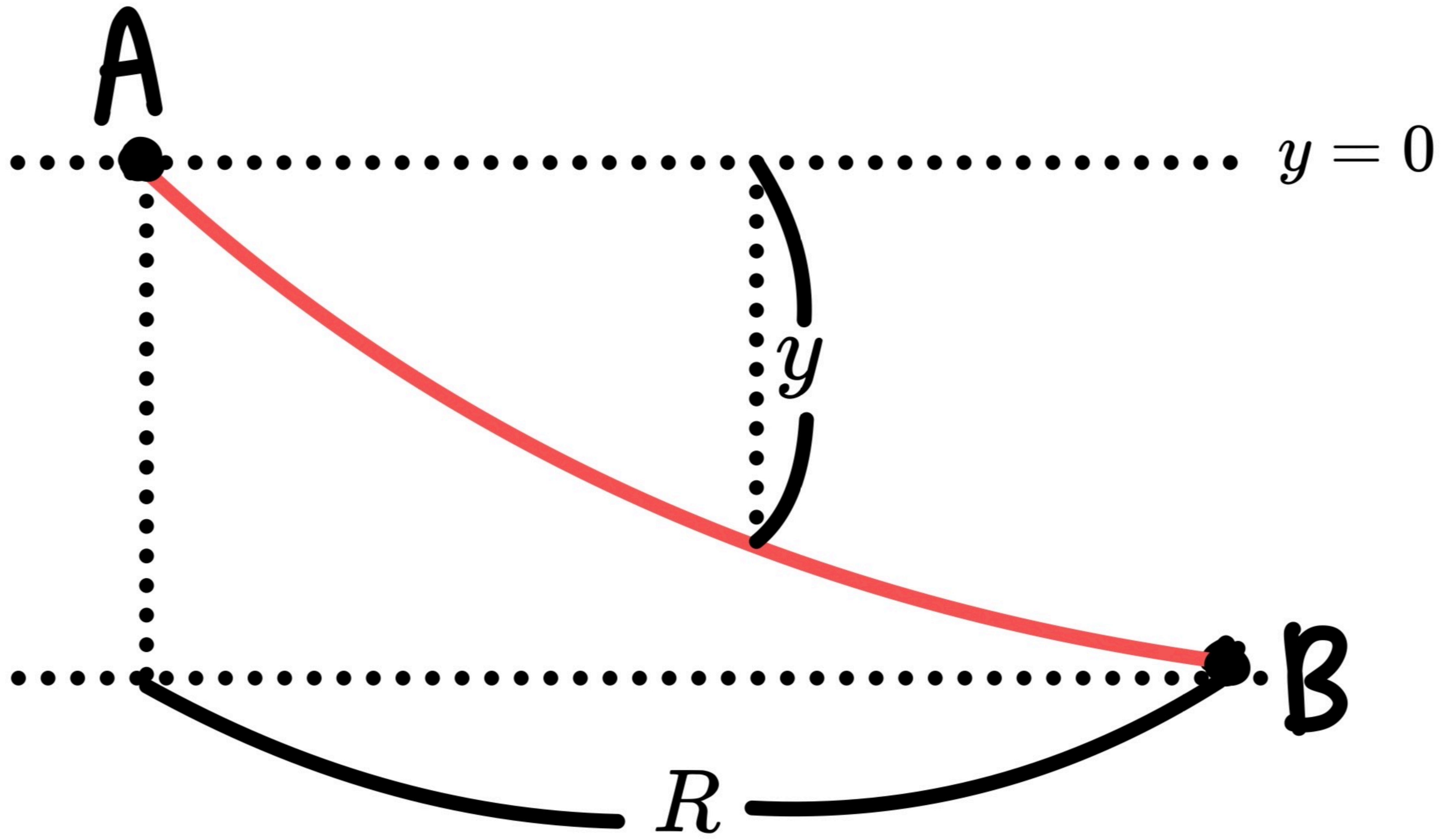
Goal: Minimize Δt

$$\frac{ds}{dt} = v \Rightarrow dt = \frac{ds}{v} \Rightarrow \Delta t = t_f - t_i = \int_{t_i}^{t_f} dt = \int \frac{ds}{v}$$

$$0 = \Delta E_{mech} = \Delta K + \Delta U_g = \frac{1}{2}mv^2 - mgy$$

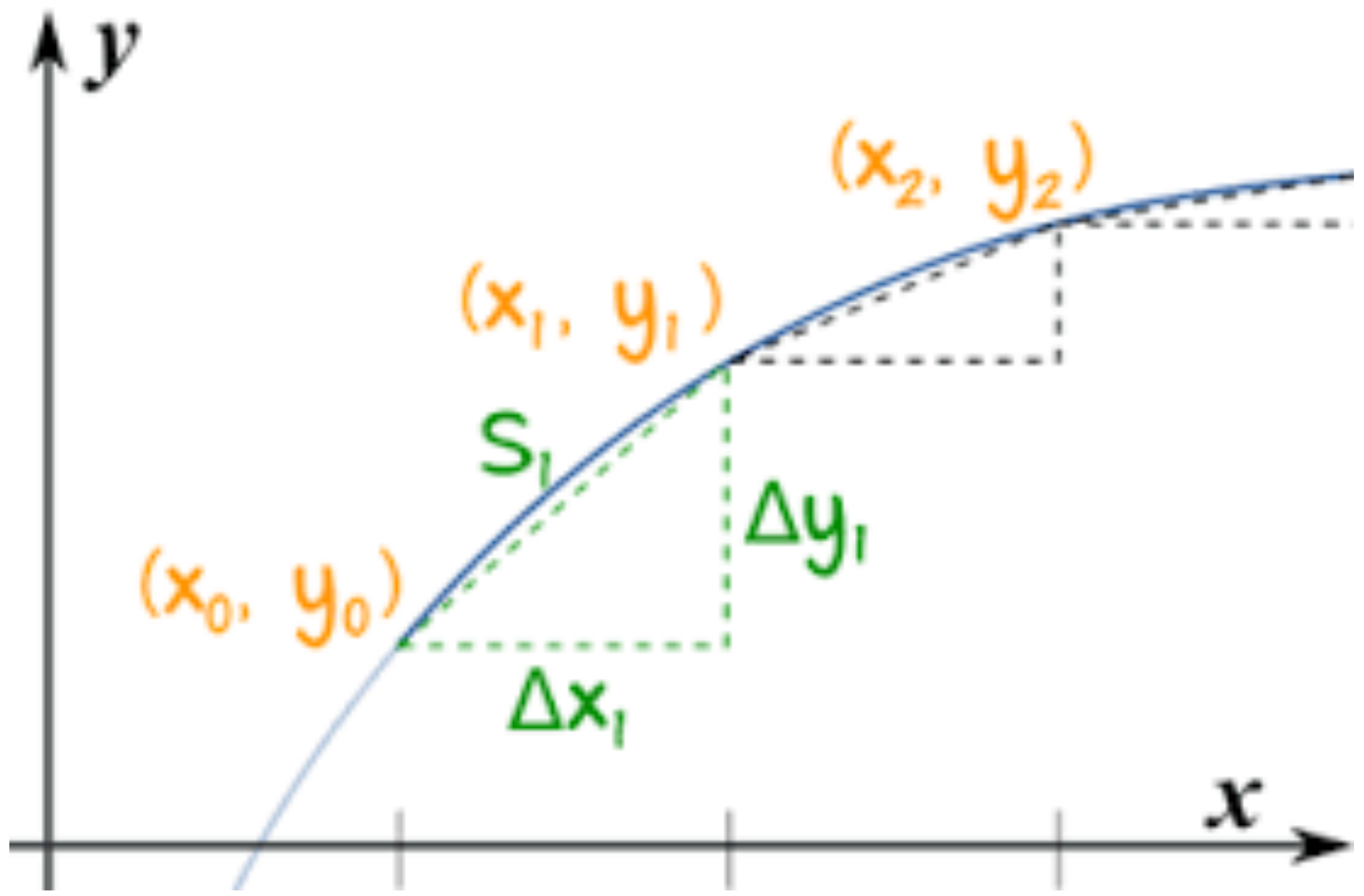
$$\Rightarrow v = \sqrt{2gy}$$

$$\Rightarrow \Delta t = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2gy}}$$



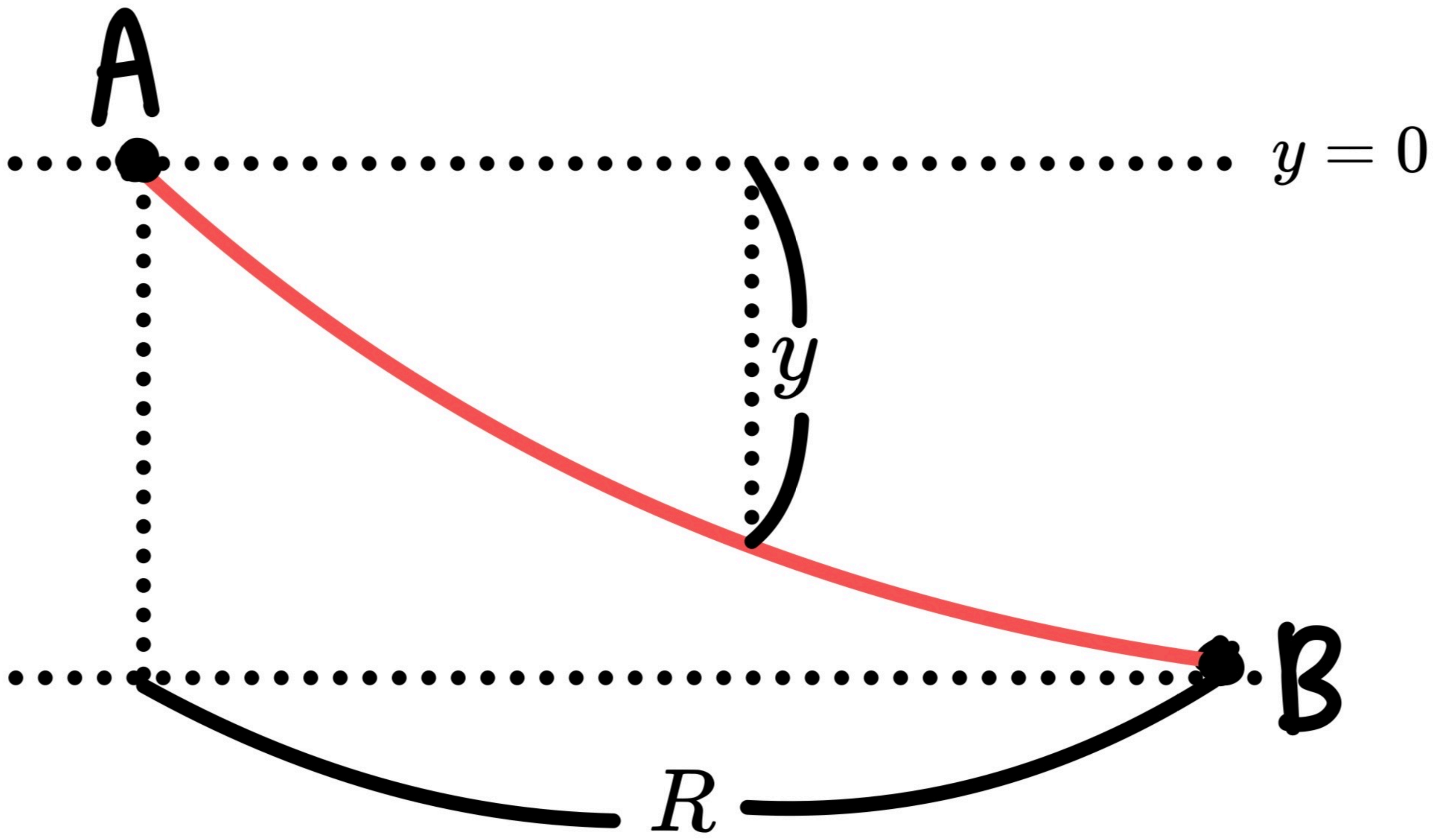
Brachistochrone

Goal: Minimize Δt



$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\Delta t = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2gy}} = \int_0^R \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}\, dx$$



The Euler-Lagrange Equation

Let $S = \{y \in C^1[a,b] | y(a) = y_a \; y(b) = y_b\}$, and let $J: S \to \mathbb{R}$ be a function of the form

$$J(y) = \int_a^b f(y(x), y'(x), x)dx$$

If J has an extremum at $y_0 \in S$, then y_0 satisfies the Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Paritial derivatives

$$f = f(x, t)$$

$$\frac{\partial f}{\partial x}(x, t) = \lim_{h \to 0} \frac{f(x + h, t) - f(x, t)}{h}$$

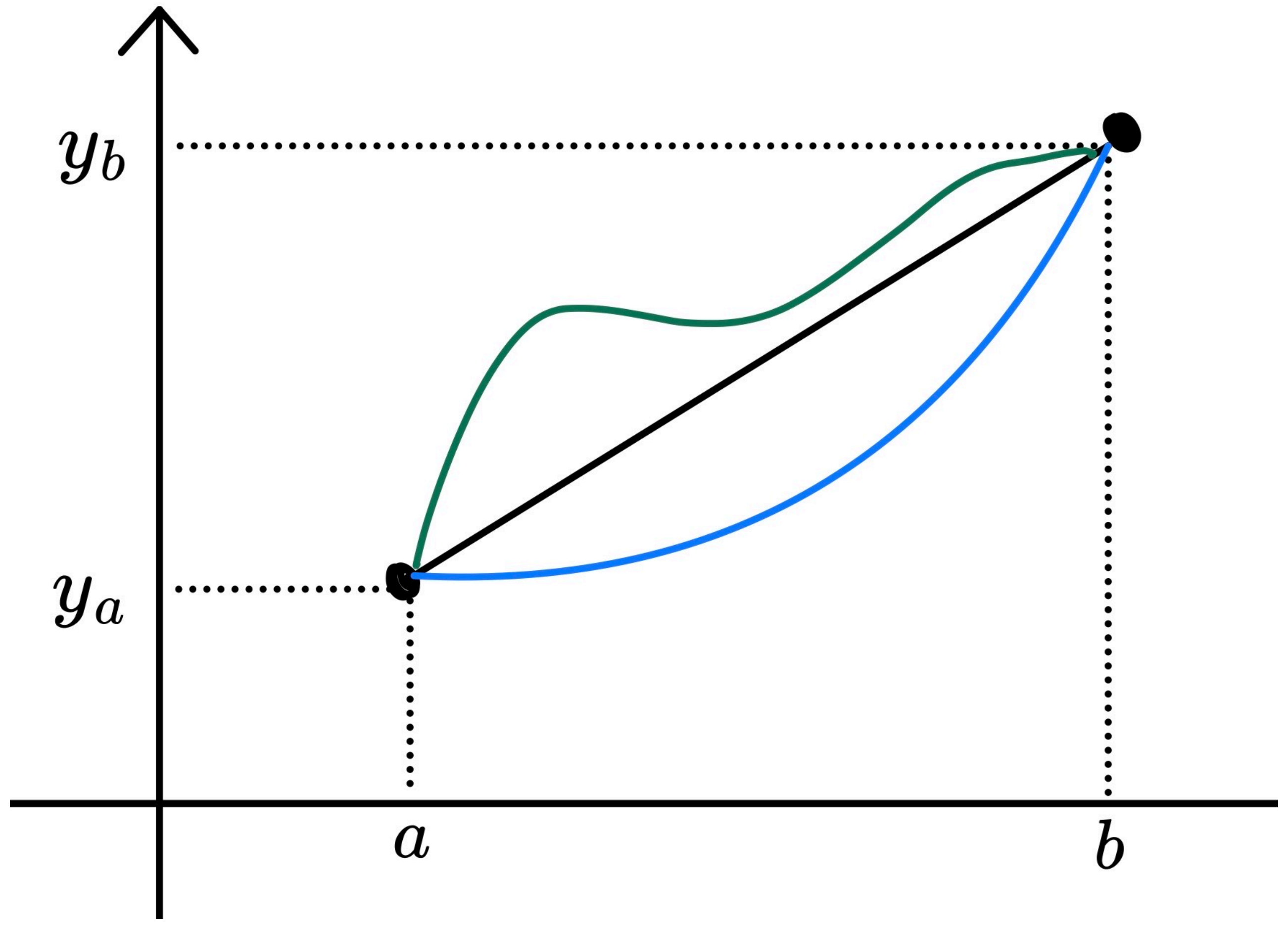
$$\frac{\partial f}{\partial t}(x, t) = \lim_{h \to 0} \frac{f(x, t + h) - f(x, t)}{h}$$

A proof of the Euler-Lagrange Equation

Suppose that there exists $y_0 \in S$ such that minimizes J.

Write $y(x)$ as $y(x,\alpha) = y_0(x) + \alpha\eta(x)$

By hypothesis, we have $\left[\frac{\partial J}{\partial \alpha}\right]_{\alpha=0} = 0$



Using $\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_a^b f(y(x), y'(x), x)dx = \int_a^b \frac{\partial f}{\partial \alpha}(y(x), y'(x), x)dx \cdots$ (Leibniz's rule)

The Beltrami Identity

If $f = f(y, y')$

If $f = f(y, y')$, then $\dfrac{\partial f}{\partial x} = 0$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Leftrightarrow f - y'\frac{\partial f}{\partial y'} = constant = C$$

$$\Delta t = \int_0^R \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\,dx = J(y) = \int_a^b f(y(x), y'(x))\,dx$$

$$f(y, y') = \frac{\sqrt{1+(y')^2}}{\sqrt{y}}$$

$$f - y'\frac{\partial f}{\partial y'} = constant = C$$

$$\frac{\partial}{\partial y'}\left(\frac{\sqrt{1+(y')^2}}{\sqrt{y}}\right) = \frac{y'}{\sqrt{y}\sqrt{1+(y')^2}}$$

$$f - y'\frac{\partial f}{\partial y'} = \frac{\sqrt{1+(y')^2}}{\sqrt{y}} - \frac{(y')^2}{\sqrt{y}\sqrt{1+(y')^2}} = C$$

$$\frac{1}{\sqrt{y}\sqrt{1+(y')^2}} = C$$

$$y' = \frac{dy}{dx} = \sqrt{\frac{1 - Cy}{Cy}}$$

$$y = \frac{1}{C}\sin^2\frac{\theta}{2} = \frac{1}{2C}(1 - \cos\theta)$$

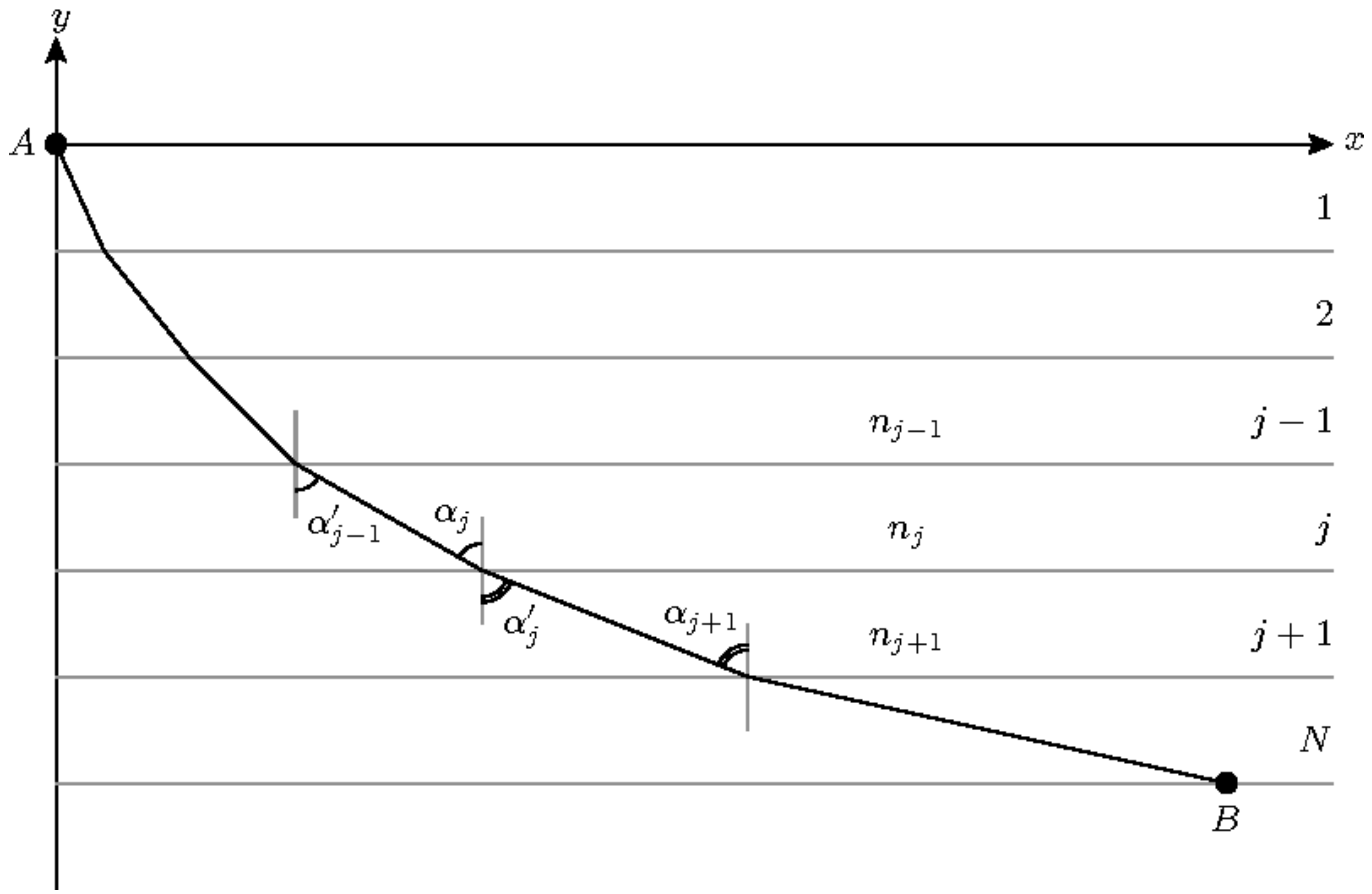
$$x = \int dx = \int \sqrt{\frac{Cy}{1-Cy}}\,dy = \int \frac{1}{C}\sin^2\frac{\theta}{2}\,d\theta$$
$$= \int \frac{1}{2C}(1 - \cos\theta)d\theta = \frac{1}{2C}(\theta - \sin\theta)$$

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

$$x = \frac{1}{2C}(\theta - \sin\theta)$$
$$y = \frac{1}{2C}(1 - \cos\theta)$$

Bernoulli's approach to the brachistochrone problem

Idea: Light travels along the path that takes the shortest time.



Snell's Law: $\dfrac{\sin\alpha_1}{v_1} = \dfrac{\sin\alpha_2}{v_2} = \cdots = \dfrac{\sin\alpha_N}{v_N} \Rightarrow \dfrac{\sin\alpha}{v} = constant$

Bernoulli's approach to the brachistochrone curve problem

Idea: Light travels along the path that takes the shortest time.

$$v = \sqrt{2gy}$$

$$\frac{\sin\alpha}{v} = constant$$

$$\frac{\sin\alpha}{\sqrt{y}} = constant$$

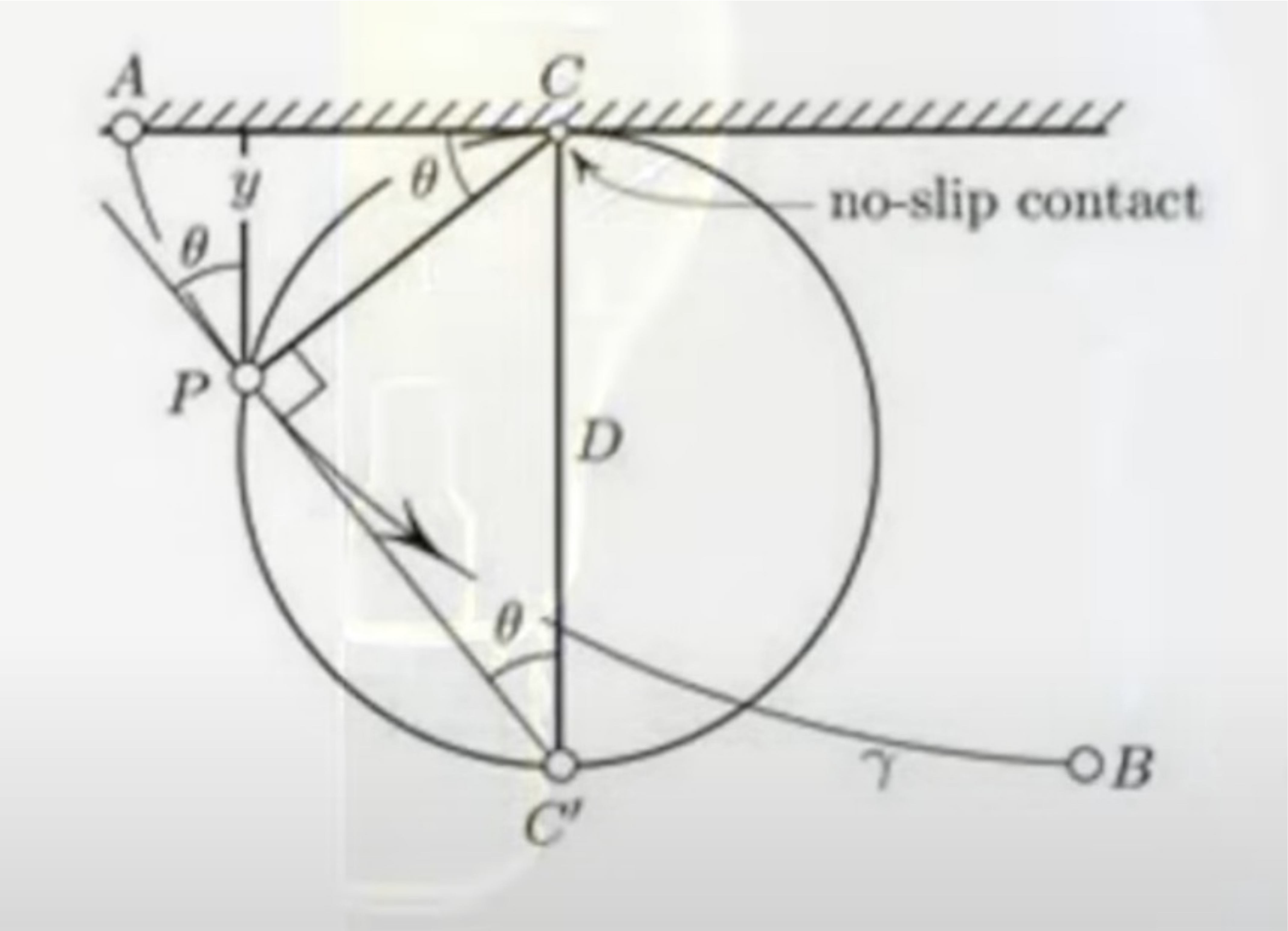
Bernoulli's approach to the brachistochrone curve problem

Idea: Light travels along the path that takes the shortest time.

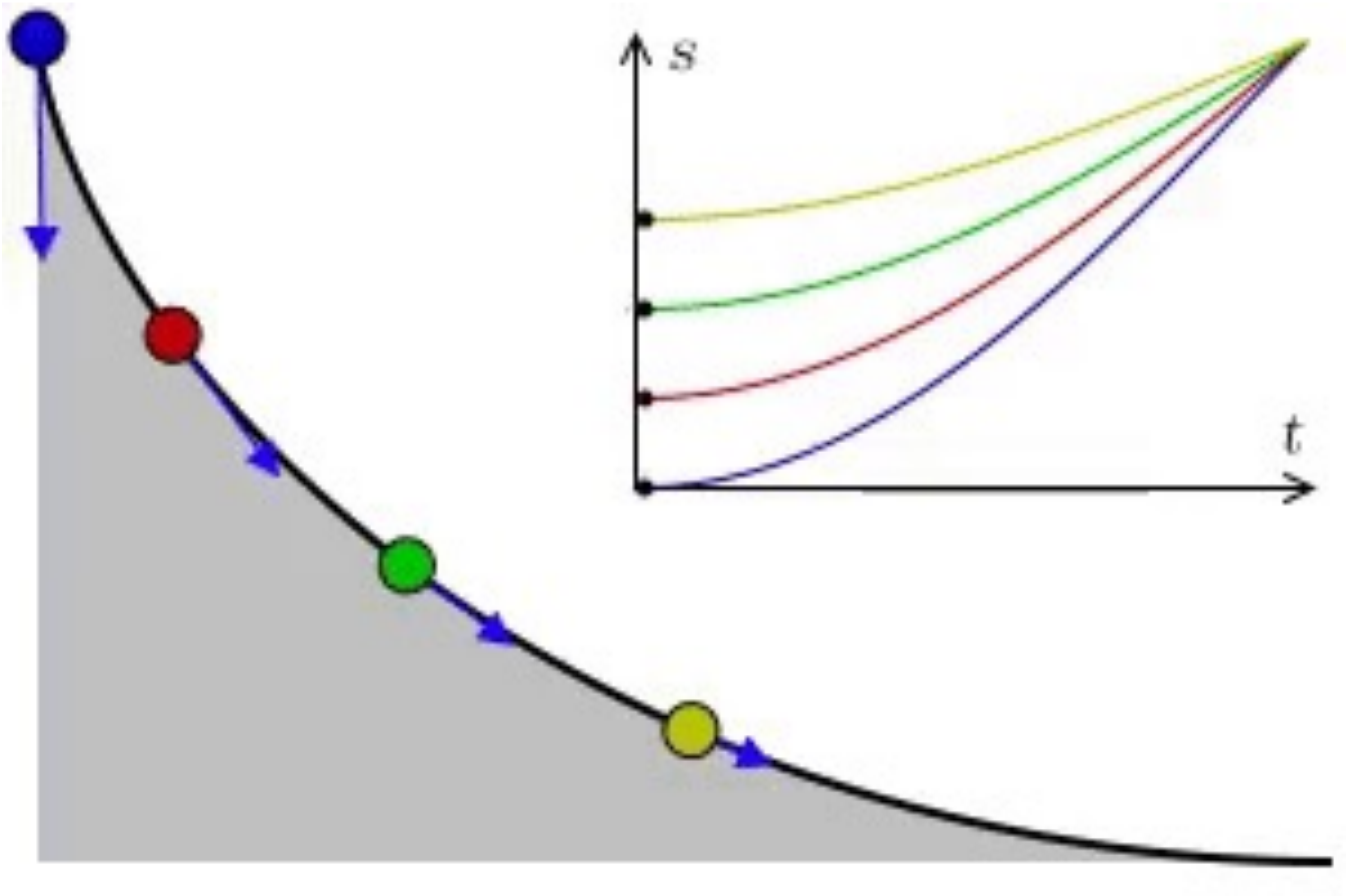
$$y = \overline{CP}\sin\theta = [D\sin\theta]\sin\theta$$

$$= D\sin^2\theta$$

$$\frac{\sin\theta}{\sqrt{y}} = constant$$



$$\Delta t = constant$$



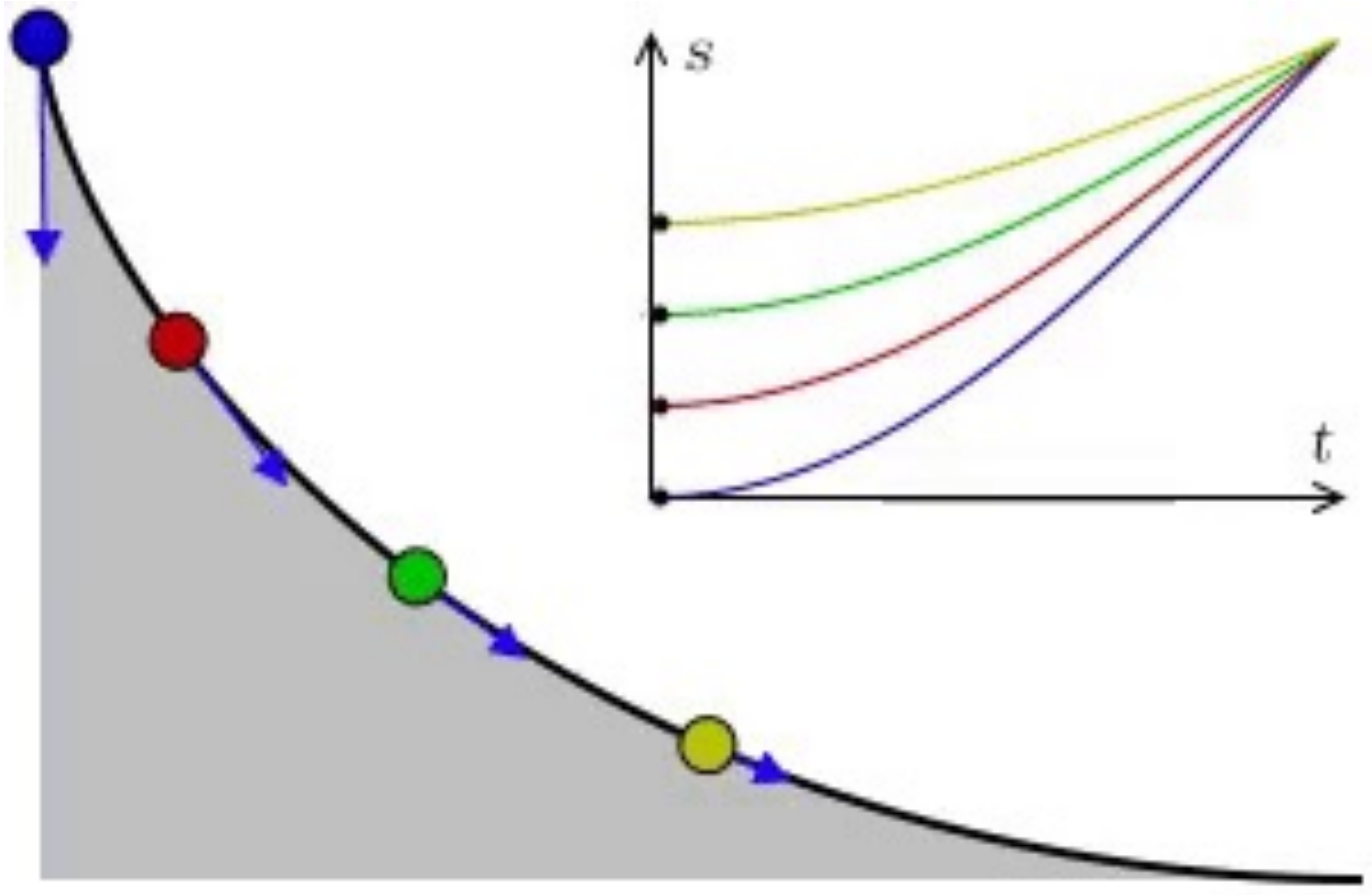
$$\Delta t = constant$$

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

$$0 = \Delta E_{mech} = \frac{1}{2}mv^2 + mg(y_{i-}y)$$

$$y_i = r(1 - \cos\theta_i), y = r(1 - \cos\theta)$$

$$v = \sqrt{2gr(\cos\theta_0 - \cos\theta)}$$



$$\Delta t = constant$$

$$dt = \frac{ds}{v} = \frac{ds}{\sqrt{2gr(\cos\theta_0 - \cos\theta)}}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \sqrt{(r(1-\cos\theta))^2 + (r\sin\theta)^2}\, d\theta$$

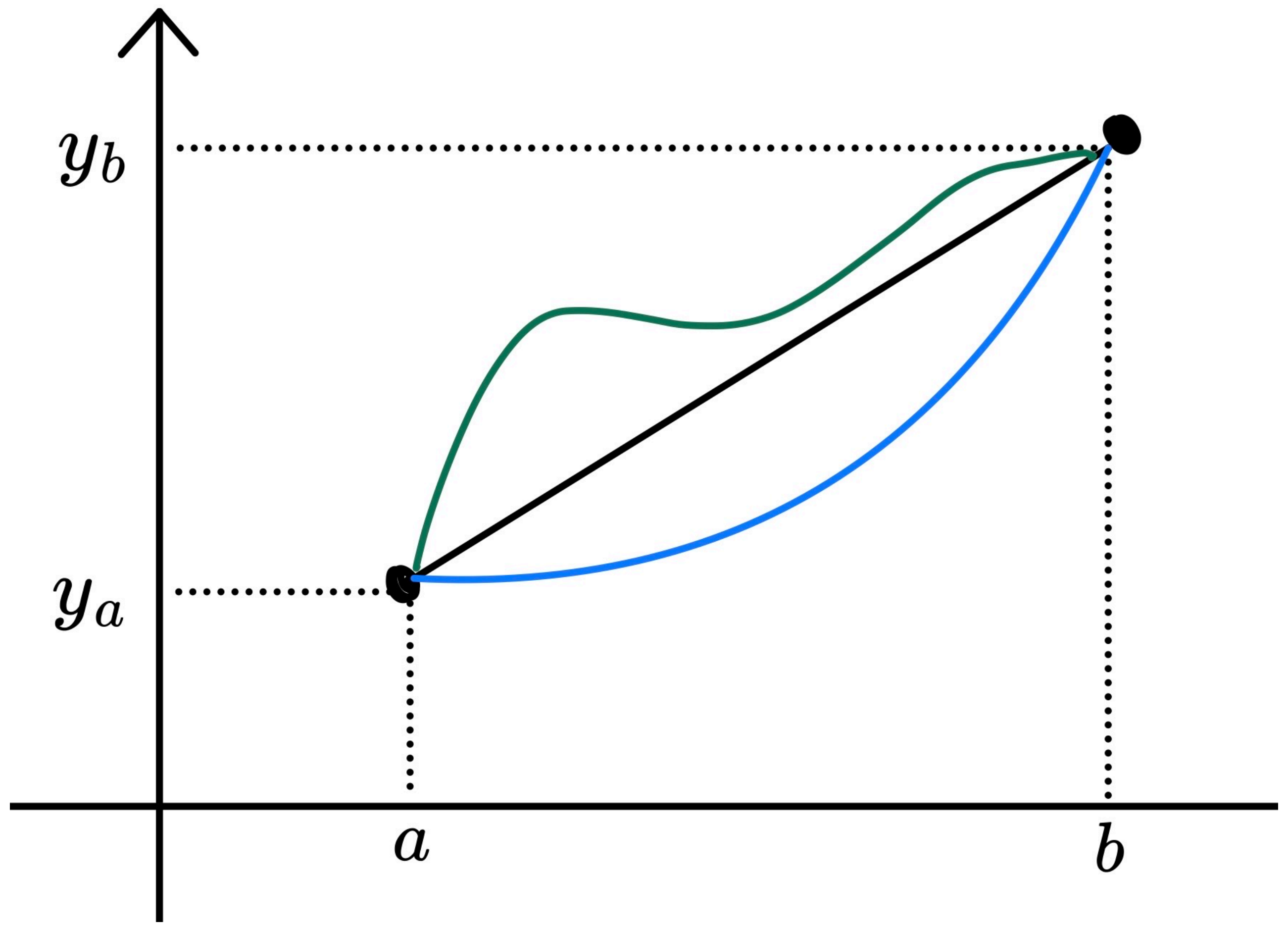
$$= r\sqrt{2}\sqrt{1-\cos\theta}\, d\theta = 2r\sin\frac{\theta}{2}\, d\theta$$

$$\Delta t = \int_{\theta_i}^{\pi} \frac{2r\sin\frac{\theta}{2}\, d\theta}{\sqrt{2gr(\cos\theta_0 - \cos\theta)}} = \int_{\theta_i}^{\pi} \sqrt{\frac{r}{g}} \frac{\sin\frac{\theta}{2}}{\cos^2\frac{\theta_i}{2} - \cos^2\frac{\theta}{2}}\, d\theta = \pi\sqrt{\frac{r}{g}}$$

The shortest curve between two points

Goal: Minimize L

$$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$f = \sqrt{1 + (y')^2} = f(y, y')$ ⇒Use the Beltrami Identity

$$f - y'\frac{\partial f}{\partial y'} = constant = C = \sqrt{1 + (y')^2} - \frac{(y')^2}{\sqrt{1 + (y')^2}} = \frac{1}{\sqrt{1 + (y')^2}}$$

The shortest curve between two points

Goal: Minimize L

$\frac{1}{\sqrt{1+(y')^2}} = constant \Rightarrow y' = constant \Rightarrow y$: a straight line