Introduction of Gamma function

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Motivation

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

Motivation

$$\left(\frac{1}{2}\right)! = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdots 2 \cdot 1???$$

Motivation

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt = (x-1)!$$

Motivation

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt = (z-1)!$$

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Euler's Infinite product

$$\left[\left(\frac{2}{1}\right)^n \frac{1}{n+1}\right]\left[\left(\frac{3}{2}\right)^n \frac{2}{n+2}\right]\left[\left(\frac{4}{3}\right)^n \frac{3}{n+3}\right]\cdots = n!$$

Infinite product

$$\left[\left(\tfrac{2}{1}\right)^n \tfrac{1}{n+1}\right] \left[\left(\tfrac{3}{2}\right)^n \tfrac{2}{n+2}\right] \left[\left(\tfrac{4}{3}\right)^n \tfrac{3}{n+3}\right] \cdots \ = n!$$

$$\lim_{m\to\infty} \frac{(m+1)^n \, m! \; n!}{(n+m)!} = \lim_{m\to\infty} \frac{(m+1)^n}{(m+1)(m+2)\cdots(m+n)} n!$$

$$\prod_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^n \frac{k}{n+k} = n!$$

Example

$$\left[\left(\frac{2}{1}\right)^{\frac{1}{2}} \frac{1}{\frac{1}{2}+1}\right] \left[\left(\frac{3}{2}\right)^{\frac{1}{2}} \frac{2}{\frac{1}{2}+2}\right] \left[\left(\frac{4}{3}\right)^{\frac{1}{2}} \frac{3}{\frac{1}{2}+3}\right] \cdots \quad =$$

$$\prod_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^{\frac{1}{2}} \frac{k}{\frac{1}{2}+k} = \frac{1}{2}!$$

Walli's product

$$\left(\frac{2}{1}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{4}{5}\right)\left(\frac{6}{5}\cdot\frac{6}{7}\right)\cdots=\prod_{k=1}^{\infty}\frac{4k^2}{4k^2-1}=\prod_{k=1}^{\infty}\frac{k^2}{k^2-\frac{1}{4}}=\frac{\pi}{2}$$

Example

$$\left(\tfrac{1}{2}\right)! = \prod_{k=1}^{\infty}\left(\frac{k+1}{k}\right)^{\frac{1}{2}}\frac{k}{\frac{1}{2}+k} = \prod_{k=2}^{\infty}\left(\frac{k}{k-1}\right)^{\frac{1}{2}}\frac{k-1}{-\frac{1}{2}+k} =$$

$$\prod_{k=2}^{\infty}\left(\frac{k}{k-1}\right)^{\frac{1}{2}}\frac{k}{-\frac{1}{2}+k}\frac{k-1}{k}\cdots \text{①}$$

$$\left(\tfrac{1}{2}\right)! = \prod_{k=1}^{\infty}\left(\frac{k+1}{k}\right)^{\frac{1}{2}}\frac{k}{\frac{1}{2}+k} = \frac{2\sqrt{2}}{3}\prod_{k=2}^{\infty}\left(\frac{k+1}{k}\right)^{\frac{1}{2}}\frac{k}{\frac{1}{2}+k}\cdots \text{②}$$

Calculating

$$① \cdot ② = \prod_{k=2}^{\infty} \frac{2\sqrt{2}}{3} \left(\frac{k+1}{k-1}\right)^{\frac{1}{2}} \frac{k-1}{k} \frac{k^2}{k^2-\frac{1}{4}}$$

$$\frac{2\sqrt{2}}{3} \lim_{m\to\infty} \sqrt{\frac{m(m+1)}{2}} \cdot \frac{1}{m+1} \cdot \prod_{k=1}^{\infty} \frac{k^2}{k^2-\frac{1}{4}} \cdot \frac{3}{4}$$

$$\frac{1}{2} \cdot \frac{\pi}{2} = \left(\frac{1}{2}!\right)^2$$

$$\therefore \frac{1}{2}! = \frac{\sqrt{\pi}}{2}$$

Euler's integral

$$\int_0^1 x^e (1-x)^n dx = \frac{1 \cdot 2 \cdots n}{(e+1)(e+2)\cdots(e+n+1)}$$

$$\int_0^1 x^e (1-x)^n \, dx$$

$$\begin{array}{ccc}
(1-x)^n & & x^e \\
 & + & \\
-n(1-x)^{n-1} & & \frac{1}{e+1} x^{e+1} \\
 & - & \\
n(n-1)(1-x)^{n-2} & & \frac{1}{(e+1)(e+2)} x^{e+2} \\
 & + & \\
\vdots & & \\
n(n-1)\cdots(2) & & \\
n! & & \frac{1}{(e+1)(e+2)\cdots(e+n)} x^{e+n}
\end{array}$$

$$\left[\frac{x^{e+1}}{e+1}(1-x)^n + \frac{n x^{e+1}}{(e+1)(e+1)}(1-x)^{n-1} + \cdots \frac{n!}{(e+1)(e+2)\cdots(e+n)} x^{e+n} \right]_0^1$$

$$= \frac{n!}{(e+1)(e+2)\cdots(e+n)}$$

Euler's integral

Let f and g continuous.

$$\int_0^1 x^{\frac{f}{g}}(1-x)^n dx = \frac{1\cdot 2\cdots n}{(f+g)(f+2g)\cdots(f+ng)} \cdot \frac{g^{n+1}}{f+(n+1)g}$$

$$\frac{1\cdot 2\cdots n}{(f+g)(f+2g)\cdots(f+ng)} = \frac{f+(n+1)g}{g^{n+1}} \int_0^1 x^{\frac{f}{g}}(1-x)^n dx$$

If $f \rightarrow 1, g \rightarrow 0$, then $n! = \int_0^1 \frac{x^{\frac{1}{g}}(1-x)^n}{g^{n+1}} dx$

Euler's integral

$$x \rightarrow x^{\frac{g}{f+g}}$$

$$dx \rightarrow \frac{g}{f+g} x^{\frac{-f}{g+f}} dx$$

$$\frac{f+(n+1)g}{g^{n+1}} \int_0^1 x^{\frac{f}{g}} (1-x)^n dx \rightarrow \frac{f+(n+1)g}{g^{n+1}} \int_0^1 \frac{g}{f+g} \left(1 - x^{\frac{g}{f+g}}\right)^n dx$$

Again $f \rightarrow 1, g \rightarrow 0$,then

$$n! = \int_0^1 \left(\frac{1-x^g}{g}\right)^n dx$$

Euler's integral

$$\lim_{g\to 0}\frac{1-x^g}{g}=\lim_{g\to 0}\frac{-lnx\cdot x^g dg}{dg}=-lnx$$

$$n!=\int_0^1 \left(-lnx\right)^n dx$$

If $x=e^{-t}, dx=-e^{-t}dt$

$$n!=\int_\infty^0 t^n\cdot -e^{-t}dt=\int_0^\infty t^n e^{-t}dt$$

$$(n-1)!=\int_0^\infty t^{n-1}e^{-t}dt=\Gamma\left(n\right)$$

Gamma function

$$\Gamma\left(n\right) = \int_0^\infty t^{n-1} e^{-t} dt = \left(n-1\right)!$$

Example

$$\Gamma\left(1\right) = \int_0^\infty t^{1-1} e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 0!$$

$$\Gamma\left(2\right) = \int_0^\infty t^{2-1} e^{-t} dt = \left[-\left(t+1\right) e^{-t}\right]_0^\infty = 1!$$

$$\Gamma\left(3\right) = \int_0^\infty t^{3-1} e^{-t} dt = \left[-\left(t^2+2t+2\right) e^{-t}\right]_0^\infty = 2!$$

$$\Gamma\left(4\right) = \int_0^\infty t^{3-1} e^{-t} dt = \left[-\left(t^3+3t^2+6t+6\right) e^{-t}\right]_0^\infty = 3!$$

Integration with parts and induction

$$\Gamma\left(n+1\right)=\int_0^\infty t^n e^{-t}dt=\left[-t^n e^{-t}\right]_0^\infty\ +\int_0^\infty nt^{n-1}e^{-t}dt=n\Gamma\left(n\right)$$

$$\Gamma\left(2\right)=1\Gamma\left(1\right)$$

Gamma function with rational numbers

$$\Gamma\left(\frac{3}{2}\right) = \int_0^\infty t^{\frac{1}{2}} e^{-t} dt = \frac{\sqrt{\pi}}{2} = \left(\frac{1}{2}\right)!$$

$$x\Gamma(x) = \Gamma(x+1)$$

$$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2}\frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{7}{2}\frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{105\sqrt{\pi}}{16} = \left(\frac{7}{2}\right)!$$

Extending domain of Gamma function

$x\Gamma(x) = \Gamma(x+1)$

If $x = 0, 0\Gamma(0) = \Gamma(1) = 0! = 1$

Gamma function is not defined on $z = 0, -1, -2, \cdots$

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Sum of powers from 1 to r

$$\sum_{0}^{\infty} r^n = \frac{1}{1-r} \ (-1 < r < 1)$$

Sum of powers from 1 to r

$$\sum_{0}^{\infty} 2^n = 1 + 2 + 4 + \cdots = \frac{1}{1-2} = -1 \text{ ??}$$

Analytic function

Definition

A function f is analytic on an open set U if $\forall z \in U$,
$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, which the series is convergent to $f(z)$ for z in a neighborhood of z_0

Analytic function

$$x^2 + x,\ \ sin\left(x\right),\ \ e^z$$

C^∞ and Analytic

C^∞ does not mean it is analytic.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x \in \mathbb{R} \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-\frac{1}{h}}}{h} = 0$$

$\frac{e^{-\frac{1}{x}}}{x^2}, \frac{e^{-\frac{1}{x}}(1-2x)}{x^4} \ldots$

C^1 is analytic in complex.

In complex function, $C^1 \to$ Analytic(Holomorphic, Entire).

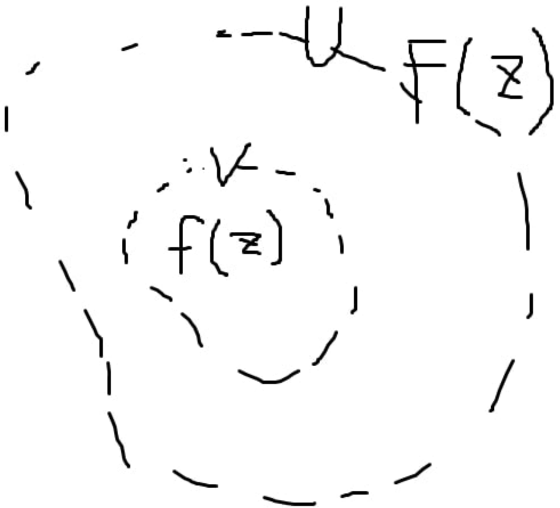
Also, sums, products, and compositions of analytic functions are analytic.

Analytic continuation

Definition

Suppose $f(z)$ is analytic in a non-empty open set V and $F(z)$ is analytic in an open set $U \supset V$. If $\forall z \in V$, $F(z) = f(z)$, then we say that F is an analytic continuation of f.

Complex plane Gamma function



Analytic continuation with sum of powers

$$\sum_{0}^{\infty} r^n = \frac{1}{1-r} \ (-1 < r < 1)$$

$$\downarrow$$

$$\sum_{0}^{\infty} r^n \to \frac{1}{1-r} \ (r \neq 1)$$

Uniqueness of analytic continuation

Theorem

If $F_1 : U \to \mathbb{C}$ and $F_2 : U \to \mathbb{C}$ are two analytic continuations of $f : V \to \mathbb{C}$, then $\forall z \in U, F_1(z) = F_2(z)$

Proof : By identity theorem

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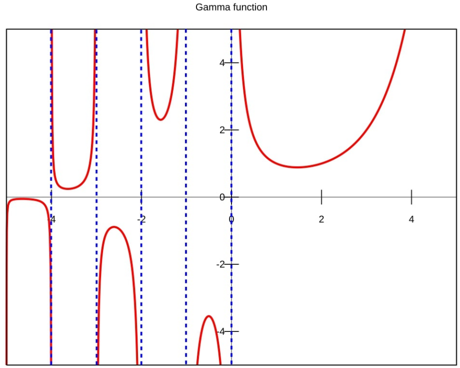
Extending domain of Gamma function

Using $x\Gamma(x) = \Gamma(x+1)$, we can expand the domain from $(1,2)$ to $\mathbb{R}$ by using analytic continuation.

Real line Gamma function



Complex numbers in Gamma function

$$z = a + bi$$

$$\Gamma\left(a + bi\right) = \int_0^\infty t^{a-1+bi} e^{-t} dt = \int_0^\infty t^{a-1} e^{-t} t^{bi} dt$$

$$e^{iz} = cos\left(z\right) + i \cdot sin\left(z\right)$$

$$e^{i\,lnz} = cos\left(lnz\right) + i \cdot sin\left(lnz\right) = z^i$$

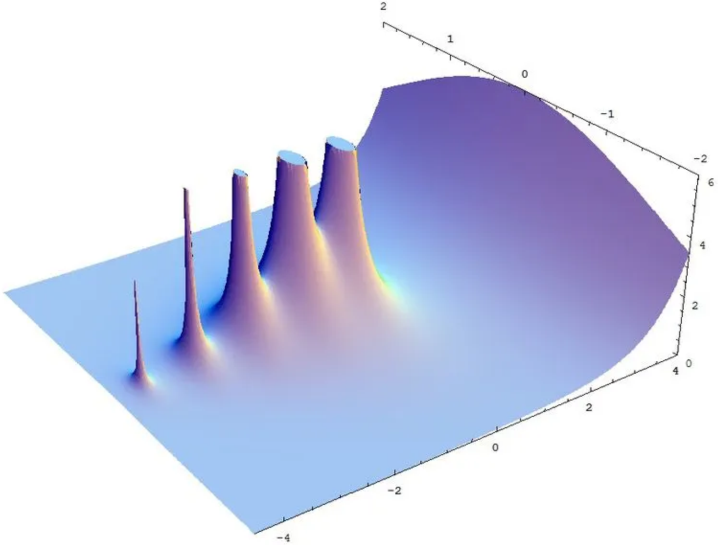
Complex numbers in Gamma function

$$\Gamma(1+i) = \int_0^\infty t^i e^{-t} dt = \int_0^\infty (cos(lnt) + i \cdot sin(lnt)) e^{-t} dt$$

$$\Gamma(1+i) \approx 0.49 - 0.15i$$

$$\Gamma(2+i) = (1+i)\Gamma(1+i) \approx 0.64 + 0.34i$$

Complex plane Gamma function



Beware

$\Gamma(z)$ is not the only analytic function which acts like $n!$

Analytic continuation

Definition

Suppose $f(z)$ is analytic in a non-empty open set V and $F(z)$ is analytic in an open set $U \supset V$. If $\forall z \in V$, $F(z) = f(z)$, then we say that F is an analytic continuation of f.

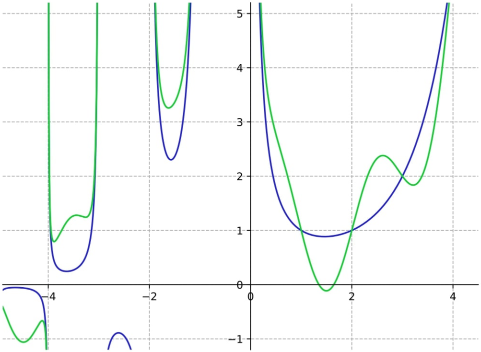
Beware

Set $\mathbb{N}$ is not open in $\mathbb{C}$, so $\Gamma(n)$ is not the analytic function of $(n-1)!$

$\Gamma(z) + ksin(m\pi z)$ is also analytic.(pseudogamma function)

But $\Gamma(z)$ is the only function which is logarithmically convex function in z of positive real numbers.
($log(f(z))$ is convex function)
(By Bohr–Mollerup theorem)

Real line with Gamma and Pseudo Gamma



Other properties

$$\overline{\Gamma(z)} = \Gamma(\overline{z})$$

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{sin(\pi z)}$$

$$\zeta(z)\Gamma(z) = \int_0^\infty \frac{t^{z-1}}{e^t-1}dt$$

Reference

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An introduction to the theory of analytic functions of one complex variable.

Thank you