Fundamental Theorem of Algebra

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Quadratic equation

We learned quadratic equation first at middle school.

Example

$$x^2 - 3x + 2 = 0$$

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$$\text{put } x = 0$$

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$$\text{put } x = 2$$

Wow

We found all the solutions!

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We found all the solutions!

But are you sure that there are only 2 roots of this equation?

F.T.A

Theorem (Fundamental Theorem of Algebra)

If there is a polynomial equation in $\mathbb{C}^1$ with n degree, there exist exactly n roots in $\mathbb{C}^1$ counted with multiplicity.

Factorization

We learned factorization.

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Then the roots are $x = 1$ or 2.

Quadratic formula

Finally, we learned how to exactly find two roots of the equation.

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example

$$6x^2 - 5x + 1 = 0$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(1)}}{2(6)}.$$

$$x = \frac{1}{2} \quad , \quad x = \frac{1}{3}$$

Minus inside a Root?

But when the number inside the root is minus, we just learned 'That does not happen' in middle school.

Imaginary number

After years past, we start to define the number $\sqrt{-1} = i$ in high school.

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For example $x^2 + 1 = 0, x = i$.

Example

The root of $x^2 - (1+i)\,x + i = (x-i)\,(x-1) = 0$ are $x = i$ and $x = 1$

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$x \longrightarrow z$

Before we start, let us now use z instead of x beacuse z is more used for complex function.

Example

$$f(z) : \mathbb{C}^1 \to \mathbb{C}^1$$

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$$iz^2 - 3z + 1$$

Example

$$f\left(z\right):\mathbb{C}^1\rightarrow\mathbb{C}^1$$

$$iz^2-3z+1$$

$$\left(1-i\right)sin\left(z\right)$$

Example

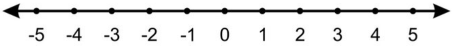
$$f(z) : \mathbb{C}^1 \to \mathbb{C}^1$$

$$iz^2 - 3z + 1$$

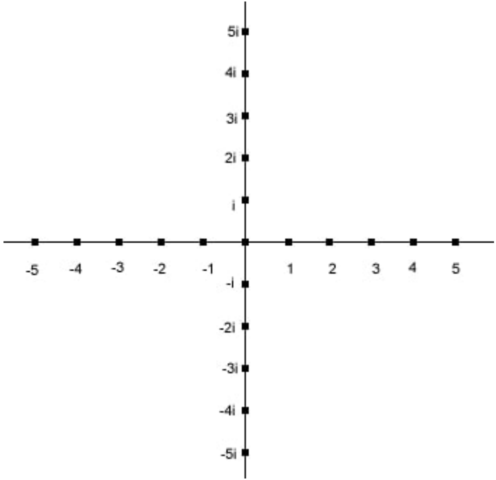
$$(1 - i)\, sin(z)$$

$$e^{iz}$$

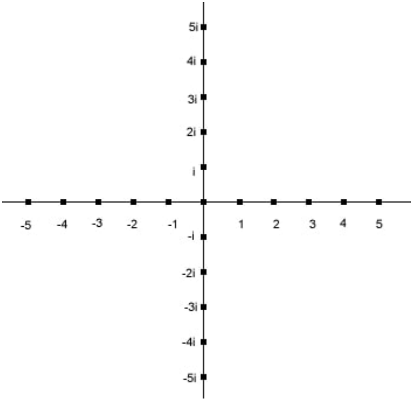
Real line



Complex plane

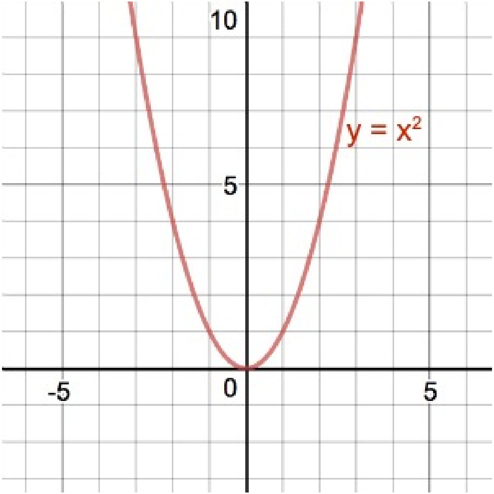


Norm of a Complex number



$$|z| = \sqrt{a^2 + b^2}$$

$y = x^2$



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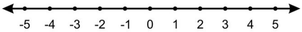
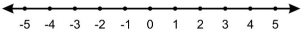
Real function

$$f : \mathbb{R}^1 \to \mathbb{R}^1$$



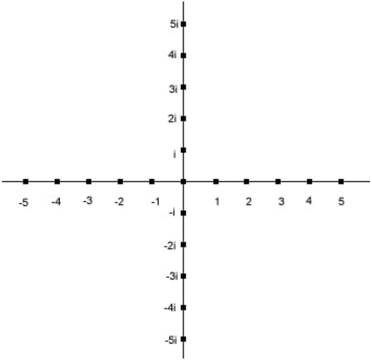
(a) x axis



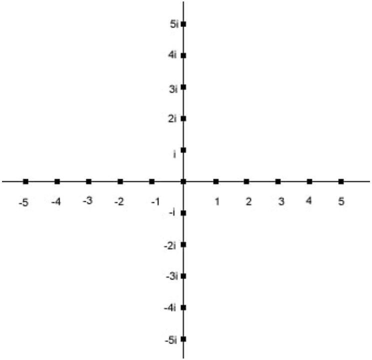
(b) y axis

Complex function

$$f : \mathbb{C}^1 \to \mathbb{C}^1$$

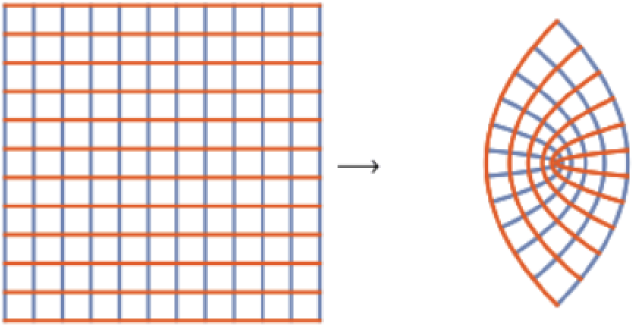


(a) Domain

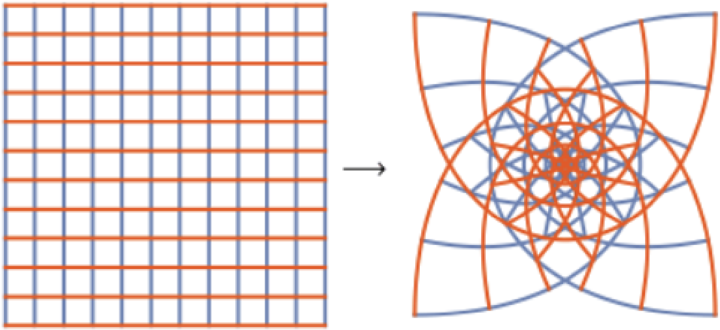


(b) Codomain

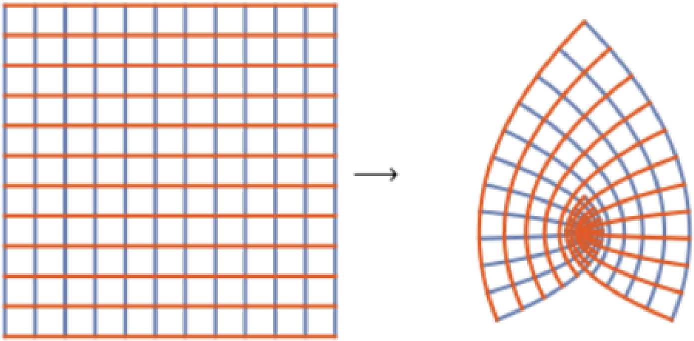
$f(z) = z^2$



$f(z) = z^3$



$f(z) = z^2 - (1+i)z + i$



Bounded function

If there exist $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all z, $f(z)$ is called a bounded function.

Example

$$f : \mathbb{R}^1 \to \mathbb{R}^1$$

$$f(x) = sin(x)$$

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$$g : \mathbb{C}^1 \to \mathbb{C}^1$$

$$h(z) = e^{iz}$$

Are polynomials bounded?

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$$f(z) = z^3 \left(1 - \frac{3}{z} + \frac{1}{z^3}\right)$$

Are polynomials bounded?

$$f : \mathbb{C}^1 \to \mathbb{C}^1$$

$$f(z) = z^3 - 3z^2 + 1$$

$$f(z) = z^3 \left(1 - \frac{3}{z} + \frac{1}{z^3}\right)$$

If $|z|$ increases very big, $\frac{1}{2}|z|^3 \leq |f(z)| \leq \frac{3}{2}|z|^3$.

Polynomials are not bounded!

Lemma (Growth Lemma)

If f is a non-constant polynomial and if a_0 is the coefficient of its leading term, then, if M is large enough,

$$|z| > M \;\rightarrow\; \frac{|a_0|}{2}|z|^n \leq |f(z)| \leq \frac{3|a_0|}{2}|z|^n.$$

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It means that complex polynomials are not bounded.

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Analytic function

Definition

Analytic function is a function that is locally given by a convergent power series.

Analytic about polynomials

f is analytic in $\mathbb{C}^1$ if it is a polynomial, and also if the equation $f = 0$ does not have a root, then $1/f$ is analytic in $\mathbb{C}^1$.

Beautiful theorem

Theorem (Liouville's theorem)

Every bounded, analytic in $\mathbb{C}^1$ function must be constant.

Example

$f : \mathbb{C}^1 \to \mathbb{C}^1$

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$f(z) = 1 + i$ is bounded, analytic in $\mathbb{C}^1$ so it is a constant.
$f(z) = z^2$ is not bounded so it is not a constant.

Our strategy

Let $f : \mathbb{C}^1 \to \mathbb{C}^1$
$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$

F.T.A

There exist at least one root of a polynomial equation.

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$$\downarrow$$

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Our aim is to prove the first one.

Proof of F.T.A

There is $f : \mathbb{C}^1 \to \mathbb{C}^1$ a polynomial with n degree.

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$\forall z \in C^1$, $f(z) \neq 0$ so $\frac{1}{f}$ is well-defined.

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$R_0 \in \mathbb{R}$ is given.
$\forall z \in \mathbb{C}^1$, if $|z| \geq R_0$, then $\exists R \in \mathbb{R}$ such that $|f(z)| \geq R$ so $0 < |\frac{1}{f(z)}| \leq |\frac{1}{R}|$.

Proof of F.T.A

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$\forall z \in C^1$, $f(z) \neq 0$ so $\frac{1}{f}$ is well-defined.
$R_0 \in \mathbb{R}$ is given.
$\forall z \in \mathbb{C}^1$, if $|z| \geq R_0$, then $\exists R \in \mathbb{R}$ such that $|f(z)| \geq R$ so $0 < |\frac{1}{f(z)}| \leq |\frac{1}{R}|$.
$\forall z \in \mathbb{C}^1$, if $|z| < R_0$, $f(z)$ is bounded and is not 0 so $\frac{1}{f(z)}$ is bounded in $\mathbb{C}^1$

Proof of F.T.A

Using the lemma , then $\frac{1}{f}$ is bounded and analytic function in $\mathbb{C}^1$ function.

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Using the lemma , then $\frac{1}{f}$ is bounded and analytic function in $\mathbb{C}^1$ function.
By 'Liouville's theorem', $\frac{1}{f}$ must be constant, but it cannot be a constant so it makes a contradiction.

Proof of F.T.A

Using the lemma , then $\frac{1}{f}$ is bounded and analytic function in $\mathbb{C}^1$ function.
By 'Liouville's theorem', $\frac{1}{f}$ must be constant, but it cannot be a constant so it makes a contradiction.
Therefore we proved 'There exist at least one root of a polynomial equation' so we are done proving F.T.A $\square$

Finding roots using F.T.A

$$z^3 - 6z^2 + 11z - 6 = 0$$

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$$(z - 1)\left(z^2 - 5z + 6\right) = 0$$

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Finding roots using F.T.A

$$z^3 - 6z^2 + 11z - 6 = 0$$
$$(z-1)\left(z^2 - 5z + 6\right) = 0$$
$$(z-1)(z-2)(z-3) = 0$$
The roots are $x = 1, 2, 3$

Complex field

It is the reason why the field of complex numbers is algebraically closed.

Try it!

There are more proofs of F.T.A.

Try it!

There are more proofs of F.T.A.
Real-analytic proofs.
Complex-analytic proofs.
Topological proofs.
Algebraic proofs by induction, from Galois Thoery.
Geometric proofs.

Reference

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Thank you