Tessellations of $\mathbb{H}^2$

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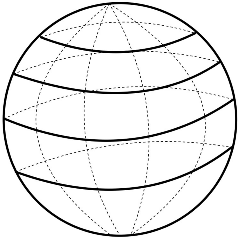
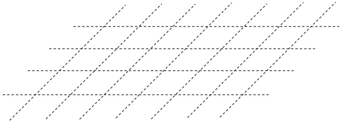
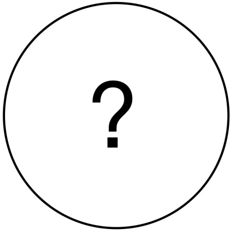
2025.09.15

Hyperbolic plane $\mathbb{H}^2$



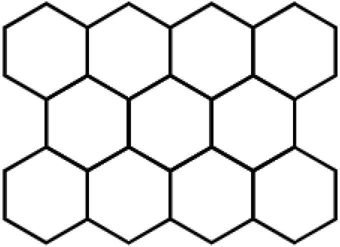


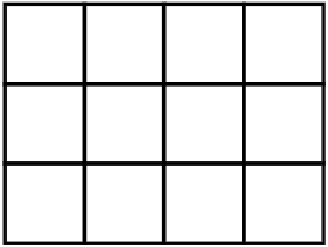


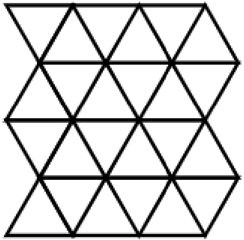
Hyperbolic plane $\mathbb{H}^2$

$\mathbb{H}^2$ vs. $\mathbb{R}^2$

Only regular triangles, squares, and hexagons tessellate $\mathbb{R}^2$

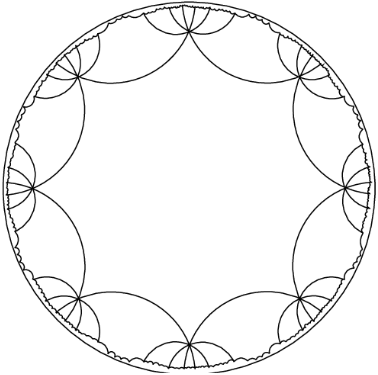


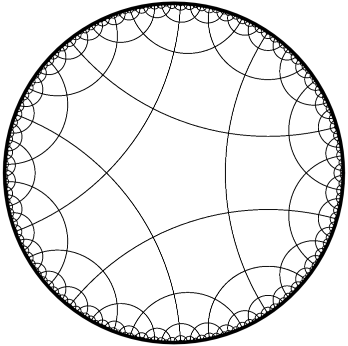




$\mathbb{H}^2$ vs. $\mathbb{R}^2$

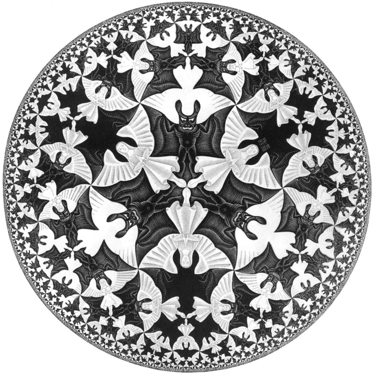
Any regular n-gon can tessellate $\mathbb{H}^2$





Outline

Which polygons tessellate $\mathbb{H}^2$?

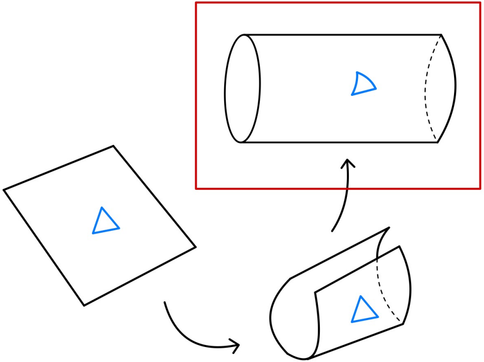


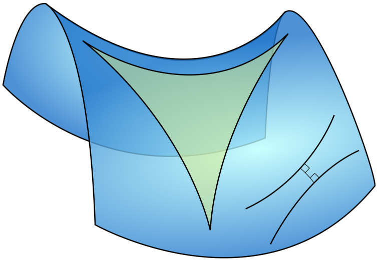
Contents

Hyperbolic plane

Curvature

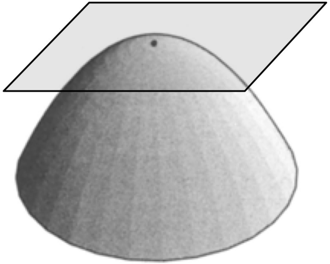
- Curvature measures how a surface deviates from the flat plane

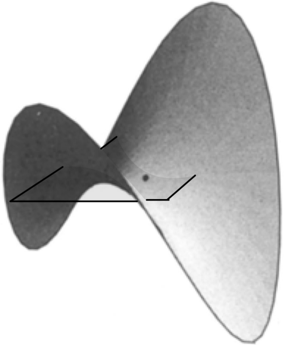




Curvature

- Curvature can be +, -, or 0



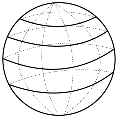


Hyperbolic plane

Remark 1.1

There exists a unique complete, simply connected surface with constant curvature 1, 0, or -1.

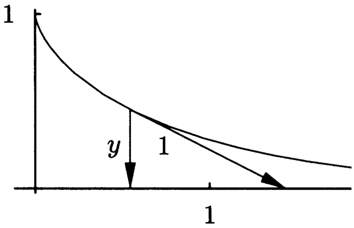
The sphere $\mathbb{S}^2$
The Euclidean plane $\mathbb{R}^2$
The hyperbolic plane $\mathbb{H}^2$

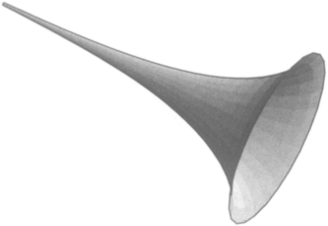


Model for $\mathbb{H}^2$

theorem 1.2 (Hilbert)

It is impossible to embed the complete $\mathbb{H}^2$ as a regular surface in $\mathbb{R}^3$



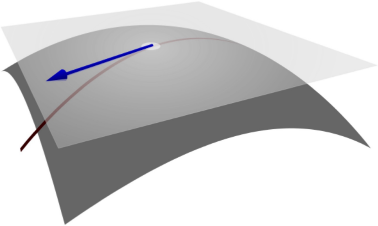


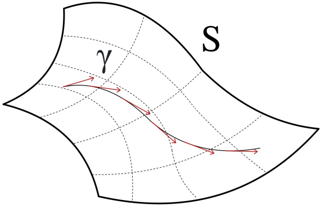
Metric

- The length of a curve $\gamma : I \to S$ is defined as

$$L(\gamma) = \int_I \|\gamma'(t)\| dt \tag{1}$$

- Euclidean norm: $\|\vec{v}\|_{Eucl} = \sqrt{v_1^2 + v_2^2}$

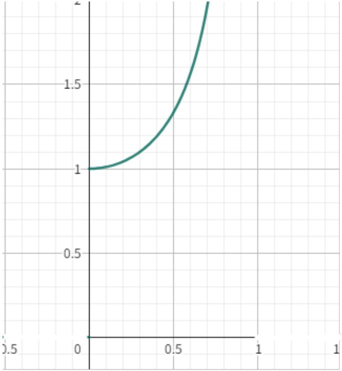


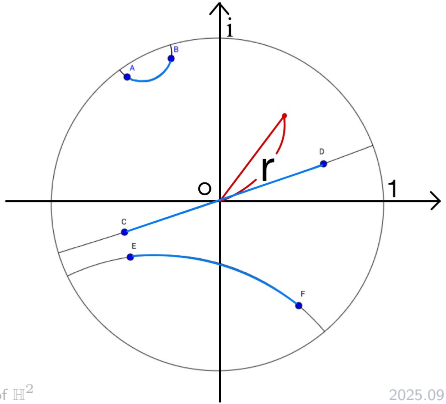


Poincaré disk model

- Assign to $\{z \in \mathbb{C} : |z| < 1\}$

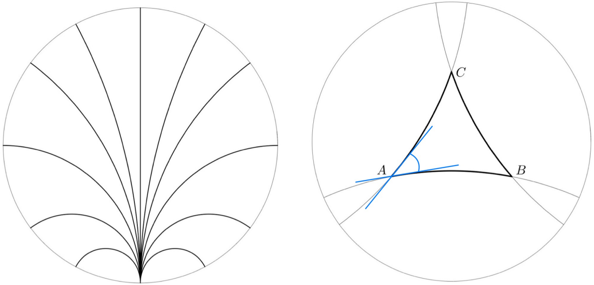
  a norm $||\vec{v}||_z = \frac{2}{1-r^2}||\vec{v}||_{Eucl}$ where $r = |z|$





Properties of Poincaré disk model

- Constant curvature -1
- Geodesics: Arc of circles, lines through the origin
- Displayed angles = hyperbolic angles



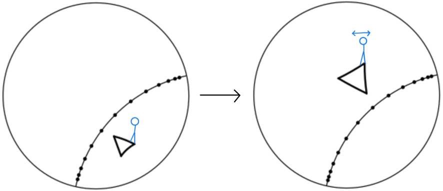
Isometry

- A metric induces a distance function; i.e. $\rho(a,b) = inf\{L(\gamma)\}$

Definition 1.3

A map $f : X \to Y$ is *an isometry* if $\rho_X(a,b) = \rho_Y(f(a), f(b))$

An isometry is a change of perspective, preserving the space



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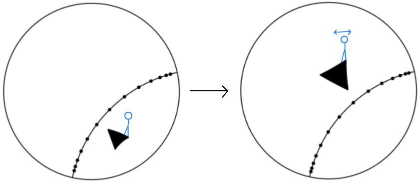

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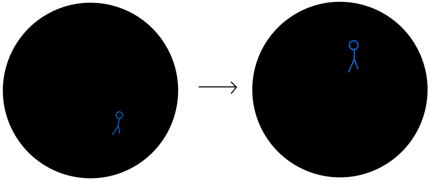
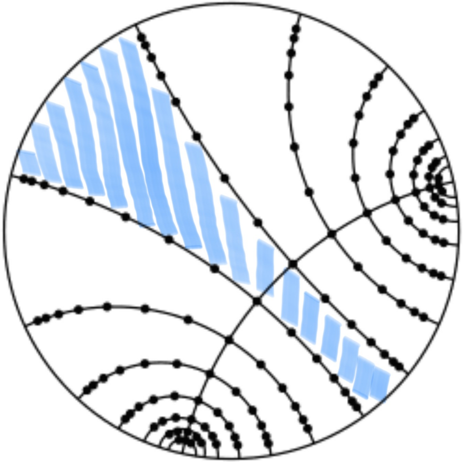


Why isometries matter?



Group of isometries

Group

Definition 2.1

Group is a set G, closed under a binary operation $*$, satisfying

1. $(ab)c = a(bc)$
2. $\exists e \in G$ s.t. $ae = ea = a$
3. $\exists a^{-1} \in G$ s.t. $aa^{-1} = a^{-1}a = e$

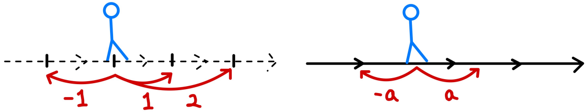
e.g. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and $\mathbb{R}$ with $+$

Group

- A group can be seen as a collection of transformations that preserve some structure

Example 2.2

$(\mathbb{R}, +)$ preserves oriented real line
$(\mathbb{Z}, +)$ preserves oriented intervals

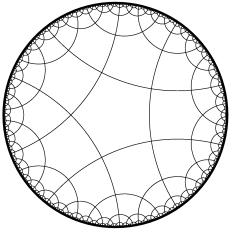


Group

Example 2.3

$\mathrm{Isom}(\mathbb{H}^2)$, under composition, is the group preserving $\mathbb{H}^2$

A group preserving a tessellation is a subgroup of $\mathrm{Isom}(\mathbb{H}^2)$

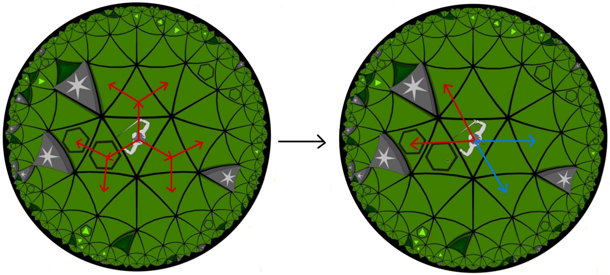


Generators of a Group

Definition 2.4

A group G is *generated* by a set $S \subseteq G$ if $\forall g \in G$, g is a finite product of elements of S and their inverses.

e.g. $(1,0)$ and $(0,1)$ generate $\mathbb{Z}^2$

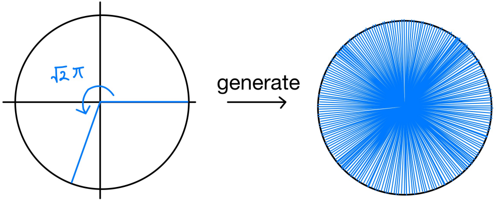


Discrete group

definition 2.5

A topological group is discrete if the group has no limit point

Discrete generators do not guarantee a discrete group

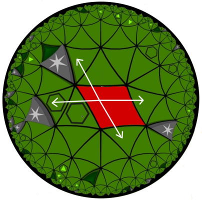


Fundamental domain

Definition 2.6

A set $D \subseteq \mathbb{H}^2$ is a *fundamental domain* for the group Γ if

1. D is a non-empty connected open set
2. $\bigcup_{\gamma\in\Gamma} \gamma(\bar{D}) = \mathbb{H}^2$
3. $D \cap \gamma(D) = \emptyset$ for all $\gamma \in \Gamma, \gamma \neq e$



Fundamental domain

Non-trivial fact 2.7

For $\Gamma \subseteq \text{Isom}(\mathbb{H}^2)$,
Γ is discrete $\Leftrightarrow$ Γ has a fundamental domain

Outline

Which polygons tessellate $\mathbb{H}^2$?

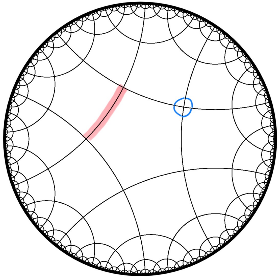
$\rightsquigarrow$ Which polygon is a fundamental domain for some Γ?

Poincaré's polygon theorem

The theorem

Theorem 3.1 (Poincaré)

Let D be a polygon.
If D has an identification and satisfies the cycle condition,
Then D is a fundamental domain for some Γ.



Identification

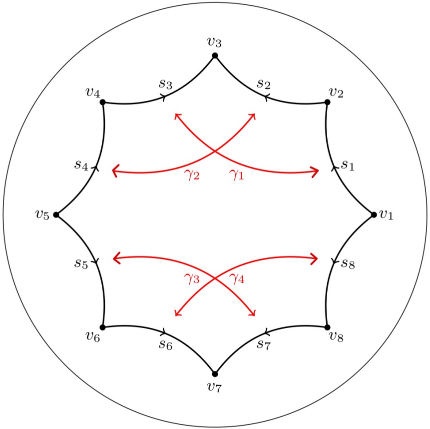
definition 3.2

An identification on a polygon D is a map from sides to isometries so that

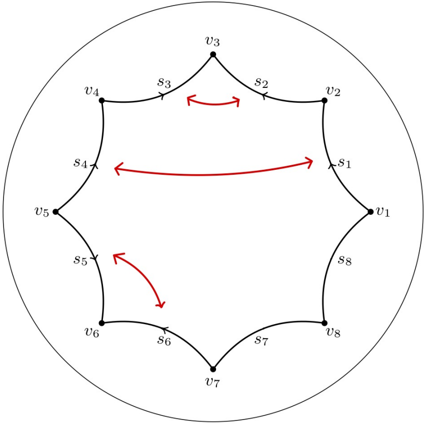
1. $\gamma_s(s) = s'$
2. $\gamma_{s'} = \gamma_s^{-1}$
3. If $s = s'$, then γ_s must be a reflection
4. For each side s, $\exists$ a neighborhood V of s so that $r_s(V \cap D) \cap D = \emptyset$

- Let us denote by Γ the group generated by the isometries

Cycle of vertices



$(v_1, v_6, v_7, v_8, v_5, v_2, v_3, v_4)$

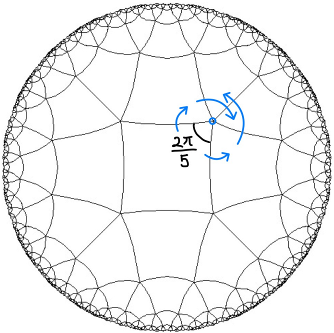


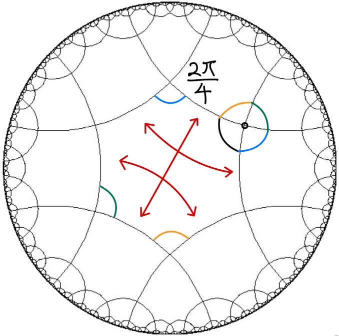
$(v_1, v_5, v_7), (v_2, v_4), (v_3), (v_6), (v_8)$

Cycle condition

Cycle condition

For each cycle $(v_1, v_2, \ldots, v_n)$, there is an integer m, so that $\sum_{i=1}^{n} \alpha(v_i) = \frac{2\pi}{m}$





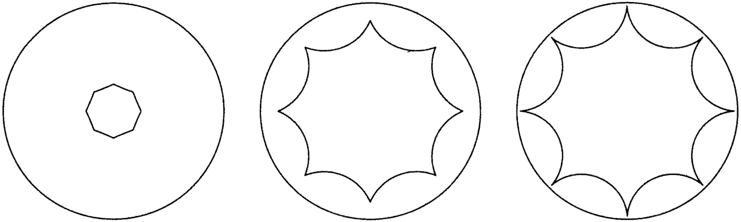
Recall

Theorem 3.1 (Poincaré)

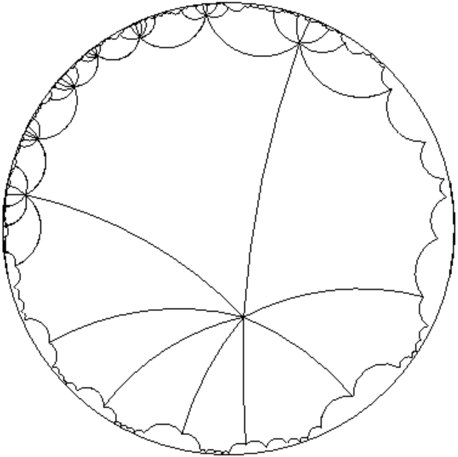
Let D be a polygon.
If D has an identification and satisfies the cycle condition,
Then the group Γ generated from identification is discrete,
D is a fundamental domain for Γ.

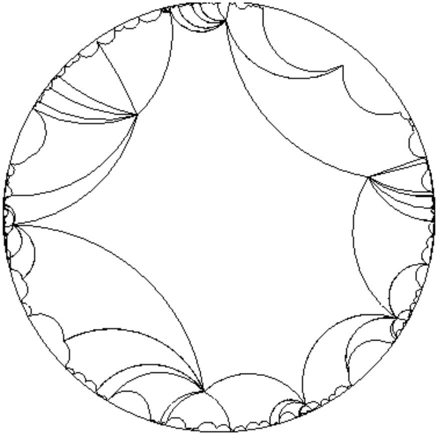
Examples(regular polygon)

- We can adjust the polygon so that the cycle condition is satisfied

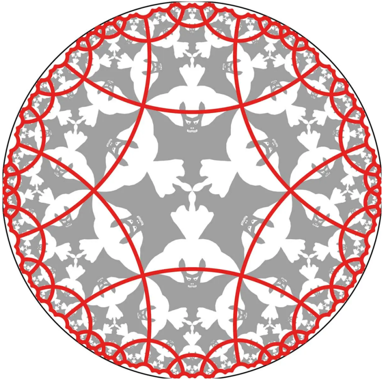


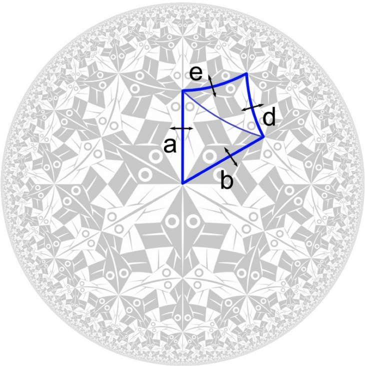
Examples(single cycle)





Examples(Escher)





Additional observation

We can investigate surfaces via tessellation

- $\mathbb{H}^2/\Gamma$ is an orbifold
- If Γ has no fixed points, then $\mathbb{H}^2/\Gamma$ is a manifold with $g \geq 2$
- Γ is a deck(covering) transformation group for $\mathbb{H}^2/\Gamma$
- $\Gamma \cong \pi_1(\mathbb{H}^2/\Gamma)$ if $\mathbb{H}^2/\Gamma$ is a manifold
- Thus, $\textbf{Teich}(S_g) \approx \textbf{Hom}(\pi_1(S_g), \textbf{Isom}^+(\mathbb{H}^2))/\textbf{Isom}(\mathbb{H}^2)$ is a collection of tessellations of 4g-gon with freely acting Γ up to isometry

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- Unsplash. Umit Canbolat
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Sketch of the Proof*

1. Paste the polygons generated by Γ; i.e. Set $X = \Gamma \times D / \sim$
2. Show that X is homeomorphic to $\mathbb{H}^2$
   - $p : X \to \mathbb{H}^2$ given by $p([(\gamma, x)]) = \gamma \cdot x$ is a covering map
   - Since $\mathbb{H}^2$ is simply connected, p is a homeomorphism