# Antipodes and Involutions

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Goal: Cover Section 2 of the paper 'Antipodes and Involution' by Benedetti and Sagan (2016)

# Plan

1. *Vector space* $\to$ *Algebra* $\to$ *Coalgebra/Bialgebra* $\to$ *Hopf algebra*
2. Takeuchi formula and why cancellations appear
3. Sign-reversing involution method (idea: split/merge)
4. **Section 2**: Polynomial Hopf algebra $\mathbb{F}[x]$: $S(x^n) = (-1)^n x^n$

# From Vector Space to Algebra

## Vector space

A linear space $V$ over a field $\mathbb{F}$.

- polynomial ring, $\mathbb{F}[x]$ with basis: $\{1, x, x^2, \dots\}$,
- polynomial ring of degree $= n$, $\mathbb{F}[x]_n$ with basis : $\{x^n\}$.
- $\mathbb{F}[x] = \bigoplus_{n \geq 0} \mathbb{F}[x]_n$.

## Algebra

$(V, m, u)$: bilinear multiplication $m : V \otimes V \to V$, unit $u : \mathbb{F} \to V$, associative with identity.

- polynomial ring $\mathbb{F}[x]$
- unit: $u(1) = 1 \in \mathbb{F}[x]$
- multiplication $m(f(x) \otimes g(x)) = f(x)g(x)$
  ex:$m(x + x^2 \otimes 2 + 3x^2) = 2x + 2x^2 + 3x^3 + 3x^4 \in \mathbb{F}[x]$

# Coalgebra and Bialgebra

## Coalgebra

$(V, \Delta, \varepsilon)$: comultiplication $\Delta : V \to V \otimes V$ (coassociative), counit $\varepsilon : V \to \mathbb{F}$.

- polynomial ring $\mathbb{F}[x]$
- counit: $\varepsilon(f(x)) = f(0) \in F[x]$
- comultiplication
  $\Delta(x^3) = \Delta(x \cdot x \cdot x) = x^3 \otimes 1 + x \otimes x^2 + x \otimes x^2 + x \otimes x^2 + x^2 \otimes x + x^2 \otimes x + x^2 \otimes x + 1 \otimes x^3 \in \mathbb{F}[x]$

## Bialgebra

$(H, m, u, \Delta, \varepsilon)$: both an algebra and a coalgebra, with $\Delta, \varepsilon$ being algebra maps.

# Hopf algebra

## Hopf algebra

Bialgebra + antipode $S : H \to H$ such that

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = u \circ \varepsilon = m \circ (S \otimes \mathrm{id}) \circ \Delta.$$

For all $c \in H$,

$$\sum_i S(c_{(1)}^{(i)}) \cdot c_{(2)}^{(i)} = \sum_i c_{(1)}^{(i)} \cdot S(c_{(2)}^{(i)}) = \varepsilon(c) \cdot 1$$

where $\Delta(c) = \sum_i c_{(1)}^{(i)} \otimes c_{(2)}^{(i)}$.

- A Hopf algebra $H$ is *graded* if it can be written as $H = \bigoplus_{n \geq 0} H_n$ so that
  1. $H_i H_j \subseteq H_{i+j}$ for all $i, j \geq 0$
  2. $\Delta H_n \subseteq \bigoplus_{i+j=n} H_i \otimes H_j$ for all $n \geq 0$, and
  3. $\varepsilon H_n = 0$ for all $n \geq 1$.
- If $H_0 \cong \mathbb{F}$, then we say that $H$ is *connected*.

# Takeuchi formula

**Theorem[Takeuchi, 1971]**

If $H$ is **graded connected** (i.e. $H = \bigoplus_{n \geq 0} H_n$, $H_0 \cong \mathbb{F}$), then it is automatically a Hopf algebra, with $S$ given by

$$S = \sum_{k \geq 0} (-1)^k \, m^{k-1} \, \pi^{\otimes k} \, \Delta^{k-1}$$

where $m^{-1} = u$, $\Delta^{-1} = \varepsilon$, and $\pi(f) = 0$ if $f$ is constant and $\pi(f) = f$ otherwise.

**Problem:** Most terms cancel.
Compute $S(x^3)$.

- $k = 0$

$$m^{-1}\Delta^{-1}(x^3) = u\varepsilon(x^3) = 0$$

# Takeuchi formula

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- $k = 1$

$$(-1)\pi(x^3) = -x^3$$

- $k = 2$

$$\begin{aligned}
&m(\pi \otimes \pi)\Delta(x^3) \\
&= m(\pi \otimes \pi)(x^3 \otimes 1 + x \otimes x^2 + x \otimes x^2 + x \otimes x^2 + x^2 \otimes x + x^2 \otimes x + x^2 \otimes x + 1 \otimes x^3) \\
&= m(x \otimes x^2 + x \otimes x^2 + x \otimes x^2 + x^2 \otimes x + x^2 \otimes x + x^2 \otimes x) \\
&= 6x^3
\end{aligned}$$

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where $m^{-1} = u$, $\Delta^{-1} = \varepsilon$, and $\pi(f) = 0$ if $f$ is constant and $\pi(f) = f$ otherwise.

- $k = 3$

$$\begin{aligned}
&(-1)m^2(\pi \otimes \pi \otimes \pi)\Delta^2(x^3) \\
&= (-1)m(\pi \otimes \pi \otimes \pi)(x^3 \otimes 1 \otimes 1 + 1 \otimes x^3 \otimes 1 + 1 \otimes 1 \otimes x^3 + x^2 \otimes x \otimes 1 + \cdots) \\
&= (-1)m(x \otimes x \otimes x + x \otimes x \otimes x + x \otimes x \otimes x + x \otimes x \otimes x + x \otimes x \otimes x + x \otimes x \otimes x) \\
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\end{aligned}$$

- $k \geq 4$

$$(-1)^k \, m^{k-1} \, \pi^{\otimes k} \, \Delta^{k-1}(x^k) = 0$$

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So,

$$S(x^3) = 0 + (-x^3) + 6x^3 + (-6x^3) + 0 + \cdots = -x^3.$$

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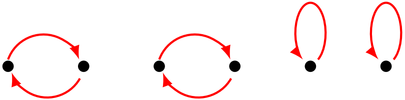
**Solution:** Use a sign-reversing involution to pair terms with opposite sign, leaving only fixed points, which yield cancellation-free formulas.

# Sign-reversing involution: split/merge idea

A map $\Phi$ is an *involution* if $\Phi \circ \Phi = id$.



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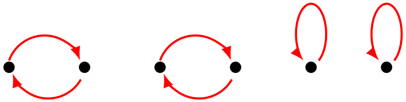

Only *fixed points* remain

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

- $(1-x)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k$
- $\binom{n}{k} \to$ #subsets of $\{1, 2, \ldots, n\}$ with $k$-elements
- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 1 + \underbrace{(-1) + \cdots + (-1)}_{\binom{n}{1}} + \underbrace{1 + \cdots + 1}_{\binom{n}{2}} \cdots = \sum_{A \subseteq \{1,2,\ldots,n\}} (-1)^{|A|}$.
- Define $\Phi : \mathcal{P}(\{1, 2, \ldots, n\}) \to \mathcal{P}(\{1, 2, \ldots, n\})$ by $\Phi(A) = \begin{cases} A \setminus \{1\} & \text{if } 1 \in A, \\ A \cup \{1\} & \text{if } 1 \notin A. \end{cases}$

# Polynomial Hopf algebra $\mathbb{F}[x]$

**Structure**

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*Ordered set partition*:
Consider $[n] := \{1, 2, \ldots, n\}$.
A set partition of $[n]$ is a collection of subsets of $[n]$, $\{B_1, B_2, \ldots, B_k\}$, with $B_i \cap B_j = \emptyset$ for any $i, j \in \{1, 2, \ldots, k\}$ and $\bigcup_{i=1}^{k} B_i = [n]$.

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Denote by $(B_1, B_2, \dots, B_k) \vDash_0 [n]$. Moreover, if $B_i \neq \emptyset$ for all $i \in \{1, 2, \dots, k\}$, then denote by $(B_1, B_2, \dots, B_k) \vDash [n]$

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$$\Delta(x) = 1 \otimes x + x \otimes 1 \to (,1) \ \& \ (1,)$$

$$(1,2) \xrightarrow{\pi\otimes\pi} (1,2), \quad (12,) \xrightarrow{\pi\otimes\pi} 0$$

# Polynomial Hopf algebra $\mathbb{F}[x]$

**Takeuchi formula**

$$S(x^n) = \sum_{k\geq 0}(-1)^k \sum_{(B_1,\dots,B_k)\vDash[n]} x^{|B_1|}\cdots x^{|B_k|} = x^n \sum_{\pi\vDash[n]}(-1)^{\text{blocks}}.$$

**Involution**

- Find the leftmost block $B_i = b_1 b_2 \dots b_j$ with $|B_l| = j \geq 2 \xrightarrow{\text{split}} b_1, b_2 b_3 \dots b_j$.
- Find the leftmost block $B_i = \{b\}$ with $b < \min B_{i+1} \xrightarrow{\text{merge}} bB_{i+1}$.
- Fixed point: $(n, n-1, \dots, 1)$ (all singletons in descending order).

## Example $n = 3$

Ordered partitions of [3]:

- $(5, 3, 249, 16, 78) \xrightarrow{\text{split}} (5, 3, 2, 49, 16, 78)$
- $(5, 3, 2, 49, 16, 78) \xrightarrow{\text{merge}} (5, 3, 249, 16, 78)$

Fixed point: $(3, 2, 1)$ only $\to (-1)^3 x^3$.

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# Theorem

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**Theorem**

In $\mathbb{F}[x]$, we have

$$S(x^n) = (-1)^n x^n.$$

# References

C. Benedetti and B. E. Sagan, *Antipodes and involutions*, arXiv:1410.5023. (This talk focuses on Sections 2–3 and their split/merge involution ideas.)