ntinuity & Differentiability

of

Weierstrass function

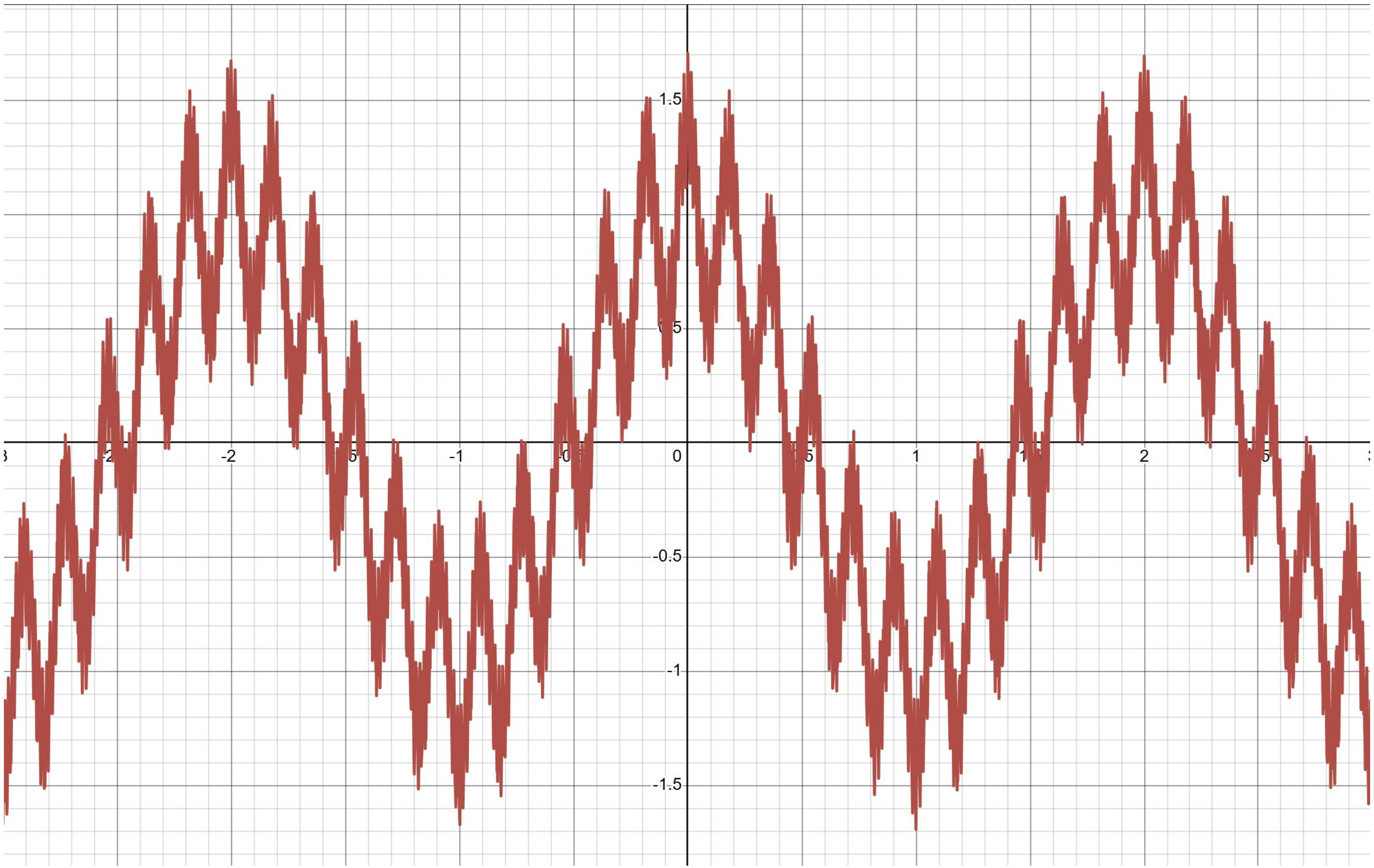
I turn with terror and horror from this **lamentable scourge** of continuous functions with no derivatives.

– Charles Hermite, letter to Thomas Joannes Stieltjes about Weierstrass functions, Correspondance d'Hermite et de Stieltjes vol.2, p.317-319

What is Weierstrass function?

$$F(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$$(0 < a < 1,\ b\text{는 양의 홀수},\ ab > 1 + \frac{3\pi}{2})$$



https://www.desmos.com/calculator?lang=ko

$a = \dfrac{5}{12}$ 이고 $b = 11$일 때, $n = 5$까지 계산한 결과

$$\mathrm{F}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$$\left(0 < a < 1,\ b\text{는 양의 홀수},\ ab > 1 + \frac{3\pi}{2}\right)$$

Definition of limit of sequence

If some sequences of functions are continuos on real number, then is its limiting functions continuos?

No!!

Because some sequences of functions can **only** be **pointwise convergence**

What is the pointwise convergence?

Let $f_n(x): X \rightarrow Y$

$$\forall x \in X, \lim_{n \to \infty} f_n(x) = f(x).$$

if and only if

$$\lim_{n \to \infty} f_n = f \text{ pointwise}$$

What is uniform convergence?

Let $\{f_n(x)\}$ be a sequence of functions defined on a domain A,

For every $\epsilon>0$, there exists a natural number N such that for all $n>N$

and for all $x\in A$,

$$|f_n(x)-f(x)|<\epsilon.$$

If this condition holds, we say that the sequence $\{f_n(x)\}$ converges uniformly to the function $f(x)$, and denote it by $f_n \rightrightarrows f$.

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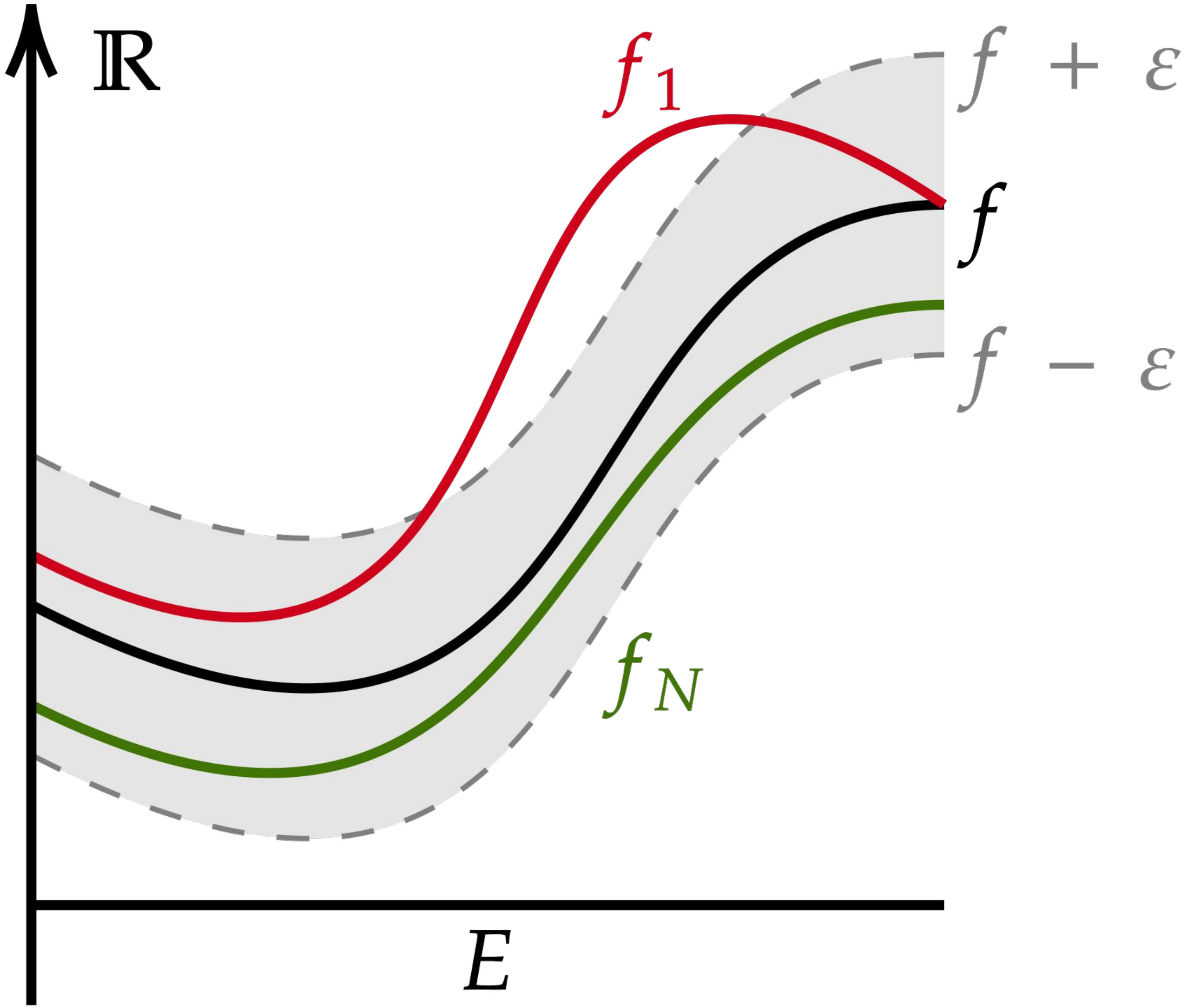

$\mathbb{R}$
f_1
$f + \varepsilon$
f
$f - \varepsilon$
f_N
E

Good properties!!

A limiting function that its sequence of function converges uniformly

has same continuity, anti-derivative, and integration

with its sequence of function.

Thus if sequence of function converges uniformly and it is continuous,

then limiting function is also continuous

Want To Show

- Thus, I want to show that Weierstrass function is uniform convergence

Weierstrass M-test

Suppose that (f_n) is a sequence of real− or complex−valued functions defined on a set A, and that there is a sequence of non−negative numbers (M_n) satisfying the conditions

- $|f_n(x)| \leq M_n$ for all $n \geq 1$ and all $x \in A$, and
- $\sum_{n=1}^{\infty} M_n$ converges.

Then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on A.

Applying M-test on the function

Let $f_n(x) = a^n \cos(b^n \pi x)$, $M_n = a^n$

Since $-1 \le \cos(b^n \pi x) \le 1$, $|f_n(x)| \le a^n = M_n$

where $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} < \infty$

Therefore Weierstrass function converge uniformly on the set $\mathbb{R}$

and so Weierstrass function is continuous

Differentiability

I would like to show there exist no x_0 contained in □

such that converge

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Differentiability

To handle it easily, make a sequence related with $\boldsymbol{x_0}$ in □

$$x_m := b^m x_0 - \alpha_m \in \left(-\frac{1}{2}, \frac{1}{2}\right] \textbf{ where } \alpha_m \in \mathbb{Z},\ m \in \mathbb{N}$$

Differentiability

To find left-hand and right-hand limit of derivative, suppose two sequence y_m and z_m such that left-hand and right-hand limit:

$$y_m < x_0 < z_m$$

$$\lim_{m \to \infty} |y_m - x_0| = 0$$

$$\lim_{m \to \infty} |z_m - x_0| = 0$$

Differentiability

$$y_m := \frac{\alpha_m - 1}{b^m} \quad \Big| \quad z_m := \frac{\alpha_m + 1}{b^m}$$

Differentiability

$$\frac{f(y_m)-f(x_0)}{y_m-x_0}=\frac{\sum_{n=0}^{\infty}a^n\cos(b^n\pi y_m)-\sum_{n=0}^{\infty}a^n\cos(b^n\pi x_0)}{y_m-x_0}$$

$$=\sum_{n=0}^{\infty}a^n\frac{\cos(b^n\pi y_m)-\cos(b^n\pi x_0)}{y_m-x_0}$$

$$=\sum_{n=0}^{m-1}a^n\frac{\cos(b^n\pi y_m)-\cos(b^n\pi x_0)}{y_m-x_0}+\sum_{n=m}^{\infty}a^n\frac{\cos(b^n\pi y_m)-\cos(b^n\pi x_0)}{y_m-x_0}$$

$$=\sum_{n=0}^{m-1}(ab)^n\frac{\cos(b^n\pi y_m)-\cos(b^n\pi x_0)}{b^n(y_m-x_0)}+\sum_{n=0}^{\infty}a^{n+m}\frac{\cos(b^{n+m}\pi y_m)-\cos(b^{n+m}\pi x_0)}{y_m-x_0}$$

Differentiability

S_1	$\sum_{n=0}^{m-1} (ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)}$
S_2	$\sum_{n=0}^{\infty} a^{n+m} \frac{\cos(b^{n+m} \pi y_m) - \cos(b^{n+m} \pi x_0)}{y_m - x_0}$

Differentiability for S_1

$$S_1 = \sum_{n=0}^{m-1} (ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)}$$

$$= \sum_{n=0}^{m-1} (ab)^n \frac{-2}{b^n (y_m - x_0)} \sin\left(\frac{b^n \pi (y_m + x_0)}{2}\right) \sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right)$$

$$= \sum_{n=0}^{m-1} -\pi \, (ab)^n \sin\left(\frac{b^n \pi (y_m + x_0)}{2}\right) \frac{\sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right)}{\frac{\pi b^n (y_m - x_0)}{2}}$$

Differentiability for S_1

$$|S_1| \leq \sum_{n=0}^{m-1} \pi \left(ab\right)^n 1 \cdot 1 = \pi \frac{\left(ab\right)^m - 1}{ab - 1} < \pi \frac{\left(ab\right)^m}{ab - 1}$$

and so for appropriate scalar ϵ_1 contained in (-1, 1) we can denote S_1 as:

$$S_1 = \epsilon_1 \frac{\pi(ab)^m}{ab - 1}$$

Returning to S_2

S_2	$\displaystyle\sum_{n=0}^{\infty} a^{n+m} \frac{\cos(b^{n+m}\pi y_m) - \cos(b^{n+m}\pi x_0)}{y_m - x_0}$

Replace it

$$y_m := \frac{\alpha_m - 1}{b^m}$$

Differentiability for S_2

Since α_{m} is integer and b is odd number,

$$\cos(b^{n+m}\pi y_m) = \cos(b^n\pi(\alpha_m - 1)) = (-1)^{b^n(\alpha_m-1)} = (-1)^{\alpha_m-1} = -(-1)^{\alpha_m}$$

By trigonometric function Sum formula

$$\begin{aligned} \cos(b^{n+m}\pi x_0) &= \cos(b^n\pi(x_m + \alpha_m)) \\ &= \cos(b^n\pi x_m)\cos(b^n\pi\alpha_m) - \sin(b^n\pi x_m)\sin(b^n\pi\alpha_m) \end{aligned}$$

Since b^n is integer, $\sin(b^n\pi x) = 0$ and so

$$= (-1)^{b^n\alpha_m}\cos(b^n\pi x_m) - 0$$

$$= (-1)^{\alpha_m}\cos(b^n\pi x_m)$$

Differentiability for S_2

Therefore S_2 could be denoted as

$$S_2 = \sum_{n=0}^{\infty} a^{n+m} \frac{\cos(b^{n+m}\pi y_m) - \cos(b^{n+m}\pi x_0)}{y_m - x_0}$$

$$= \sum_{n=0}^{\infty} a^{n+m} \frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m}\cos(b^n \pi x_m)}{y_m - x_0}$$

$$= (ab)^m (-1)^{\alpha_m} \sum_{n=0}^{\infty} a^n \frac{1 + \cos(b^n \pi x_m)}{1 + x_m}$$

Differentiability for S_2

$$\sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n \pi x_m)}{1+x_m} \geq \frac{1+\cos(\pi x_m)}{1+x_m}$$

$$\sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n \pi x_m)}{1+x_m} \geq \frac{1+\cos(\pi x_m)}{1+x_m} \geq \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

Differentiability for S_2

and so for appropriate scalar $\mu_1 > 1$ we can denote S_2 as:

$$S_2 = (ab)^m (-1)^{\alpha_m} \mu_1 \frac{2}{3}$$

Differentiability S_1+S_2

$$\frac{f(y_m)-f(x_0)}{y_m-x_0}=S_1+S_2=\epsilon_1\frac{\pi(ab)^m}{ab-1}+(ab)^m(-1)^{\alpha_m}\mu_1\frac{2}{3}$$

$$|\frac{f(y_m) - f(x_0)}{y_m - x_0}| > (ab)^m \left(\frac{2}{3} - \frac{\pi}{ab - 1} \right)$$

pathological function

$$\mathrm{T}(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}$$

Let $C([0,1],\mathbb{R})$ be a set of continuous functions $f:[0,1]\to\mathbb{R}$

The nowhere differentiable functions are most of $C([0,1],\mathbb{R})$