

The Euler-Lagrange Equation

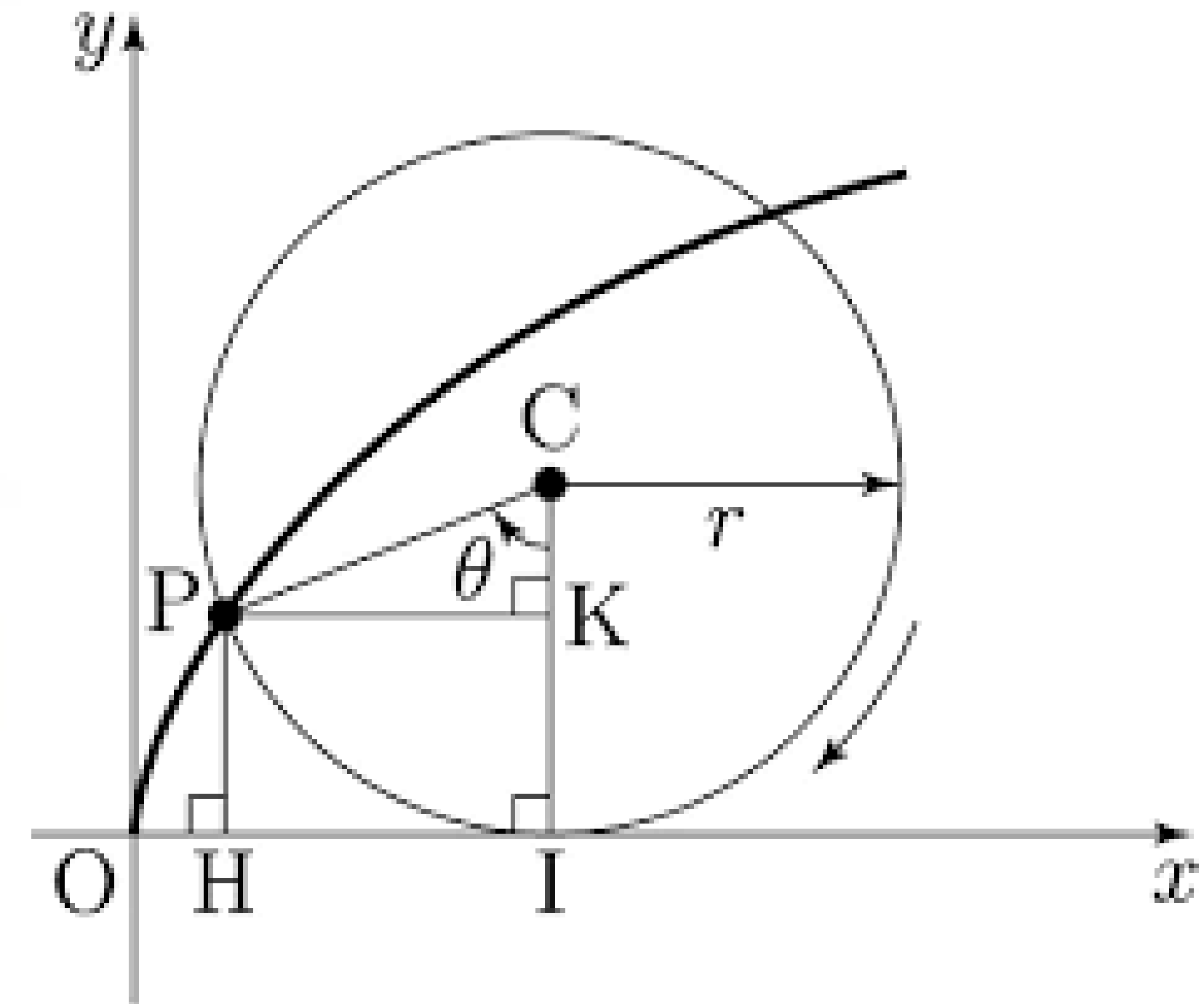
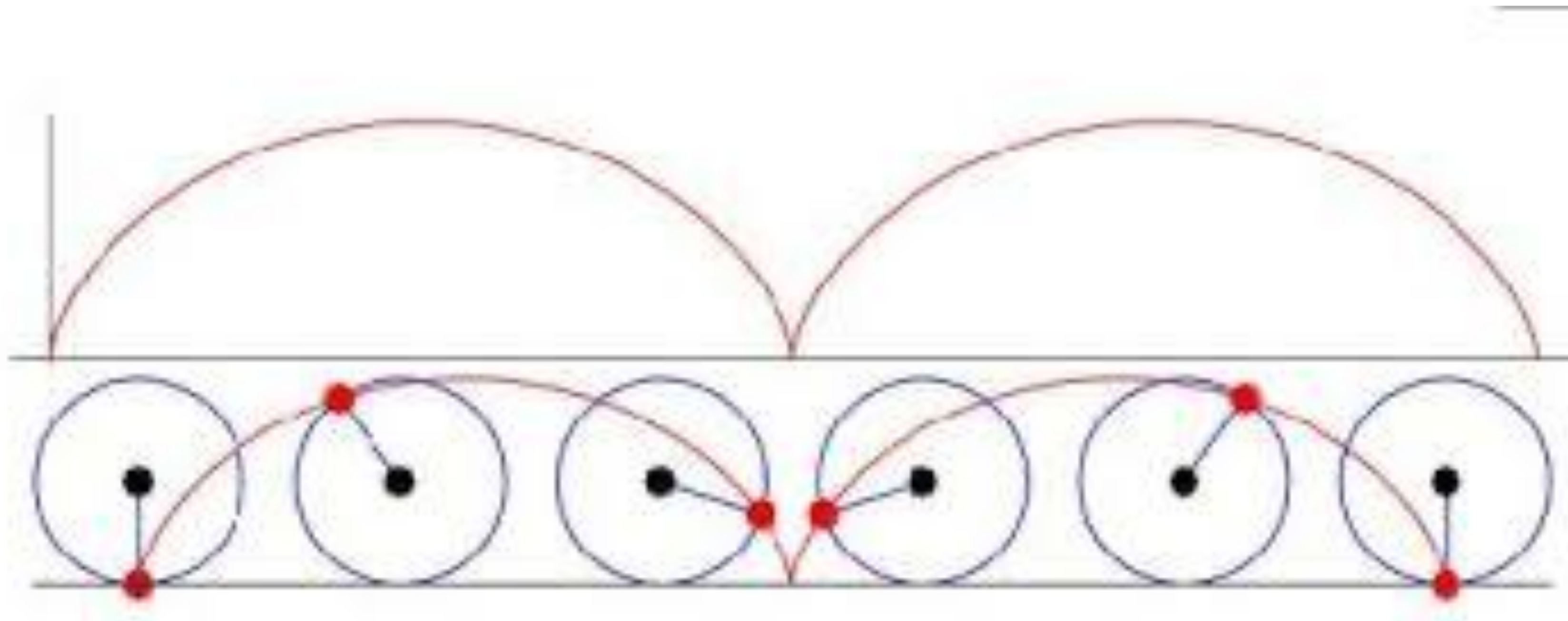
#cycloid #brachistochrone #calculus of variations

3.19.25 주현욱

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A cycloid

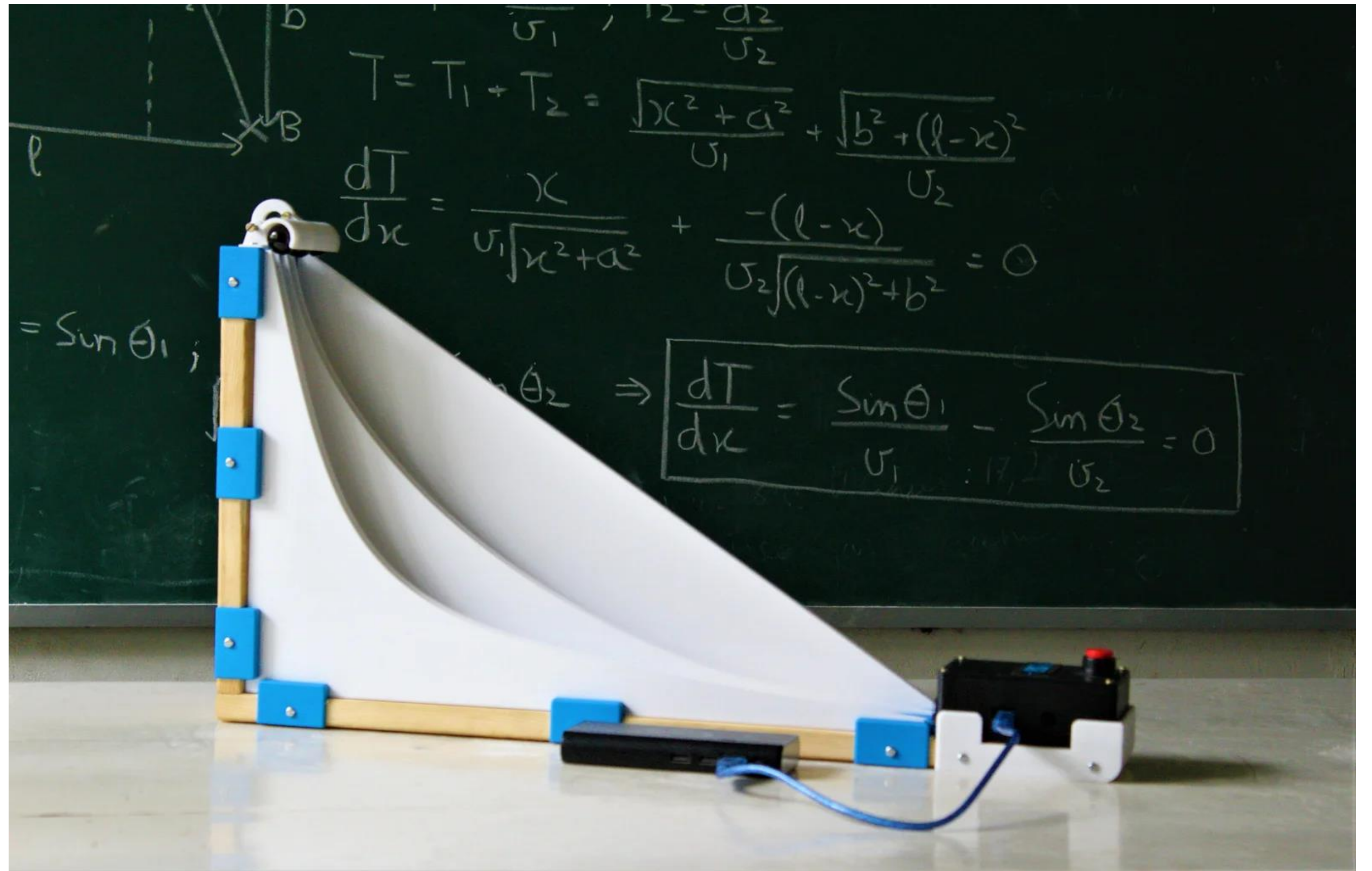


$$x = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = r - r\cos\theta = r(1 - \cos\theta)$$

Brachistochrone

The fastest route between two points

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$



Brachistochrone

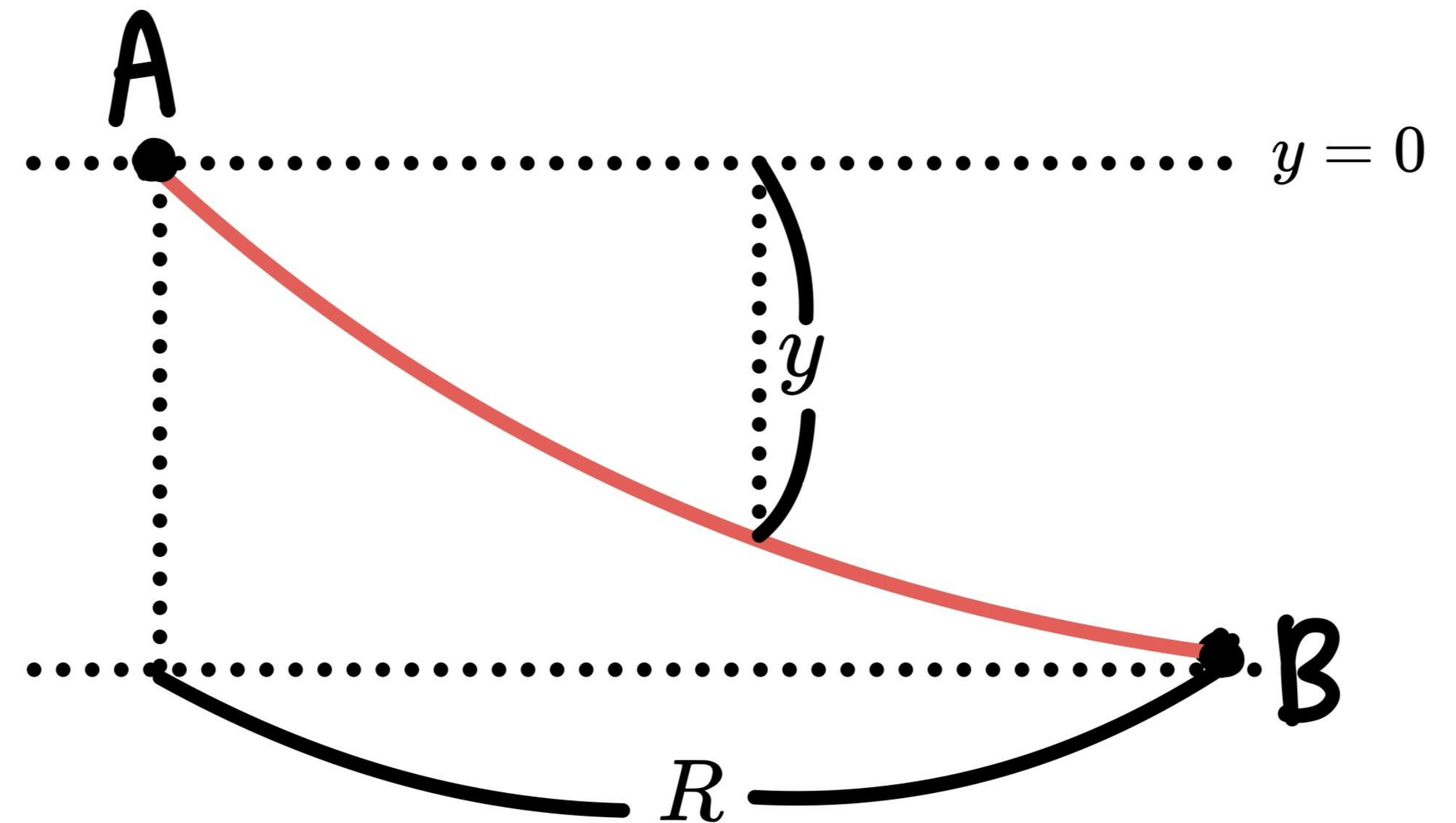
Goal: Minimize Δt

$$\frac{ds}{dt} = v \Rightarrow dt = \frac{ds}{v} \Rightarrow \Delta t = t_f - t_i = \int_{t_i}^{t_f} dt = \int \frac{ds}{v}$$

$$0 = \Delta E_{\text{mech}} = \Delta K + \Delta U_g = \frac{1}{2}mv^2 - mgy$$

$$\Rightarrow v = \sqrt{2gy}$$

$$\Rightarrow \Delta t = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2gy}}$$

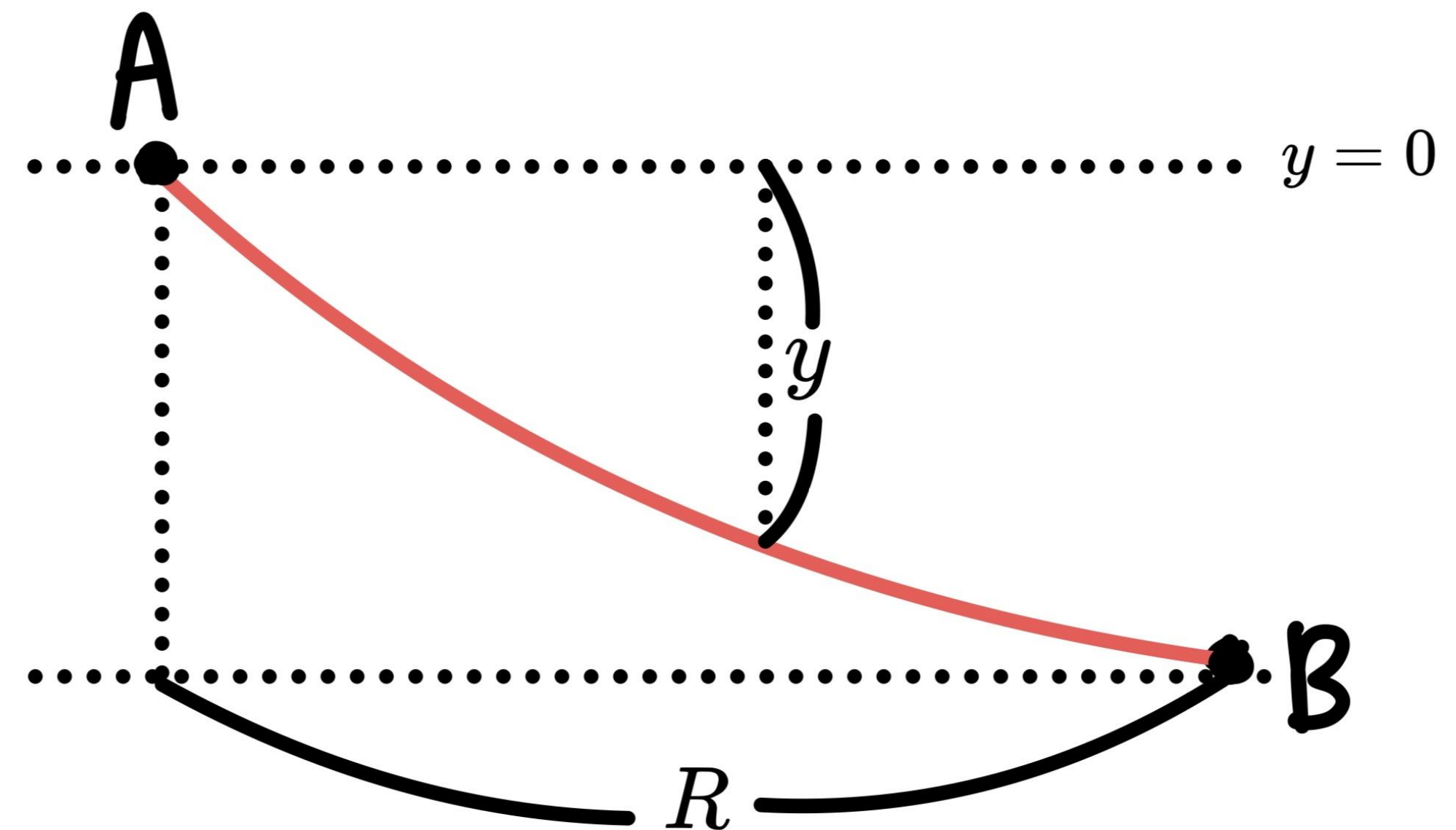
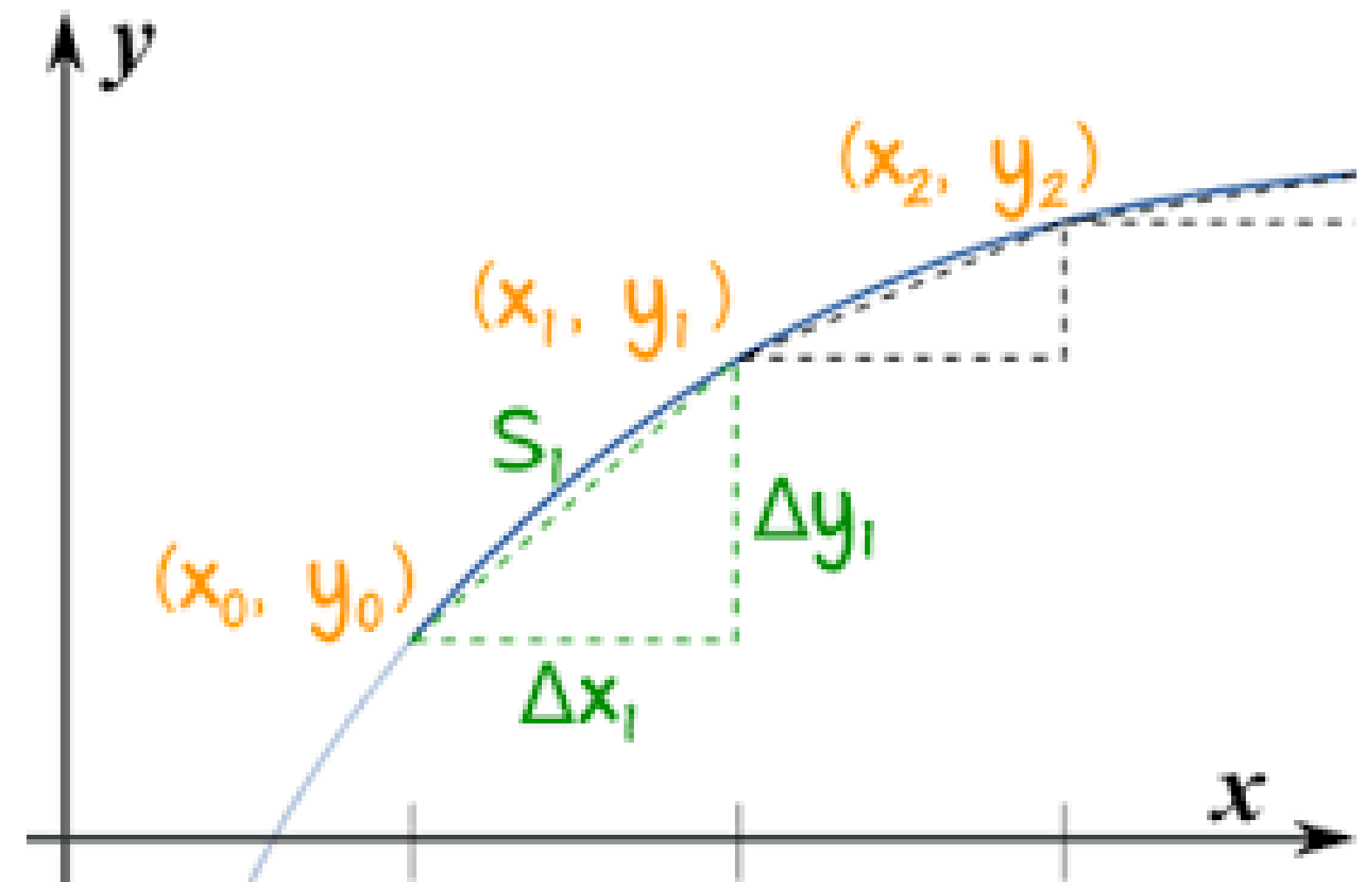


Brachistochrone

Goal: Minimize Δt

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\Delta t = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2gy}} = \int_0^R \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} dx$$



The Euler-Lagrange Equation

Let $S = \{y \in C^1[a, b] | y(a) = y_a, y(b) = y_b\}$, and let $J: S \rightarrow \mathbb{R}$ be a function of the form

$$J(y) = \int_a^b f(y(x), y'(x), x) dx$$

If J has an extremum at $y_0 \in S$, then y_0 satisfies the Euler-Lagrange Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

Partial derivatives

$$f = f(x, t)$$

$$\frac{\partial f}{\partial x}(x, t) = \lim_{h \rightarrow 0} \frac{f(x + h, t) - f(x, t)}{h}$$

$$\frac{\partial f}{\partial t}(x, t) = \lim_{h \rightarrow 0} \frac{f(x, t + h) - f(x, t)}{h}$$

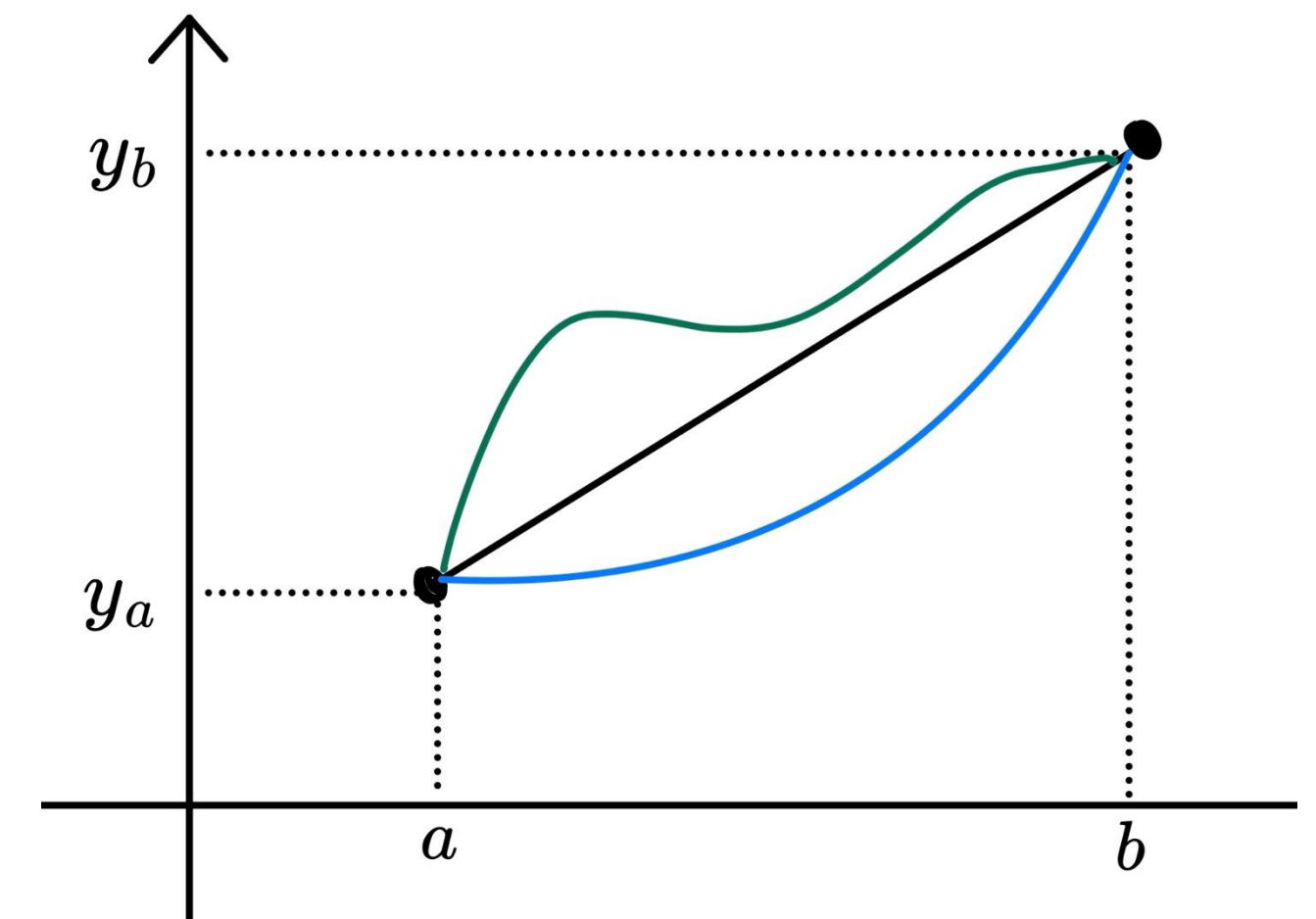
A proof of the Euler-Lagrange Equation

Suppose that there exists $y_0 \in S$ such that minimizes J .

Write $y(x)$ as $y(x, \alpha) = y_0(x) + \alpha\eta(x)$

By hypothesis, we have $\left[\frac{\partial J}{\partial \alpha}\right]_{\alpha=0} = 0$

Using $\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_a^b f(y(x), y'(x), x) dx = \int_a^b \frac{\partial f}{\partial \alpha}(y(x), y'(x), x) dx \dots$
·)Leibniz's rule)



The Beltrami Identity

If $f = f(y, y')$

If $f = f(y, y')$,then $\frac{\partial f}{\partial x} = 0$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \Leftrightarrow f - y' \frac{\partial f}{\partial y'} = \text{constant} = C$$

$$\Delta t = \int_0^R \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} dx = J(y) = \int_a^b f(y(x), y'(x)) dx$$

$$f(y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{y}}$$

$$f - y' \frac{\partial f}{\partial y'} = \textit{constant} = \mathcal{C}$$

$$\frac{\partial}{\partial y'} \left(\frac{\sqrt{1 + (y')^2}}{\sqrt{y}} \right) = \frac{y'}{\sqrt{y} \sqrt{1 + (y')^2}}$$

$$f - y' \frac{\partial f}{\partial y'} = \frac{\sqrt{1 + (y')^2}}{\sqrt{y}} - \frac{(y')^2}{\sqrt{y} \sqrt{1 + (y')^2}} = C$$

$$\frac{1}{\sqrt{y} \sqrt{1 + (y')^2}} = C$$

$$y' = \frac{dy}{dx} = \sqrt{\frac{1 - Cy}{Cy}}$$

$$y = \frac{1}{C} \sin^2 \frac{\theta}{2} = \frac{1}{2C} (1 - \cos \theta)$$

$$\begin{aligned} x &= \int dx = \int \sqrt{\frac{Cy}{1-Cy}} dy = \int \frac{1}{C} \sin^2 \frac{\theta}{2} d\theta \\ &= \int \frac{1}{2C} (1 - \cos \theta) d\theta = \frac{1}{2C} (\theta - \sin \theta) \end{aligned}$$

$$x = r(\theta - \sin\theta)$$

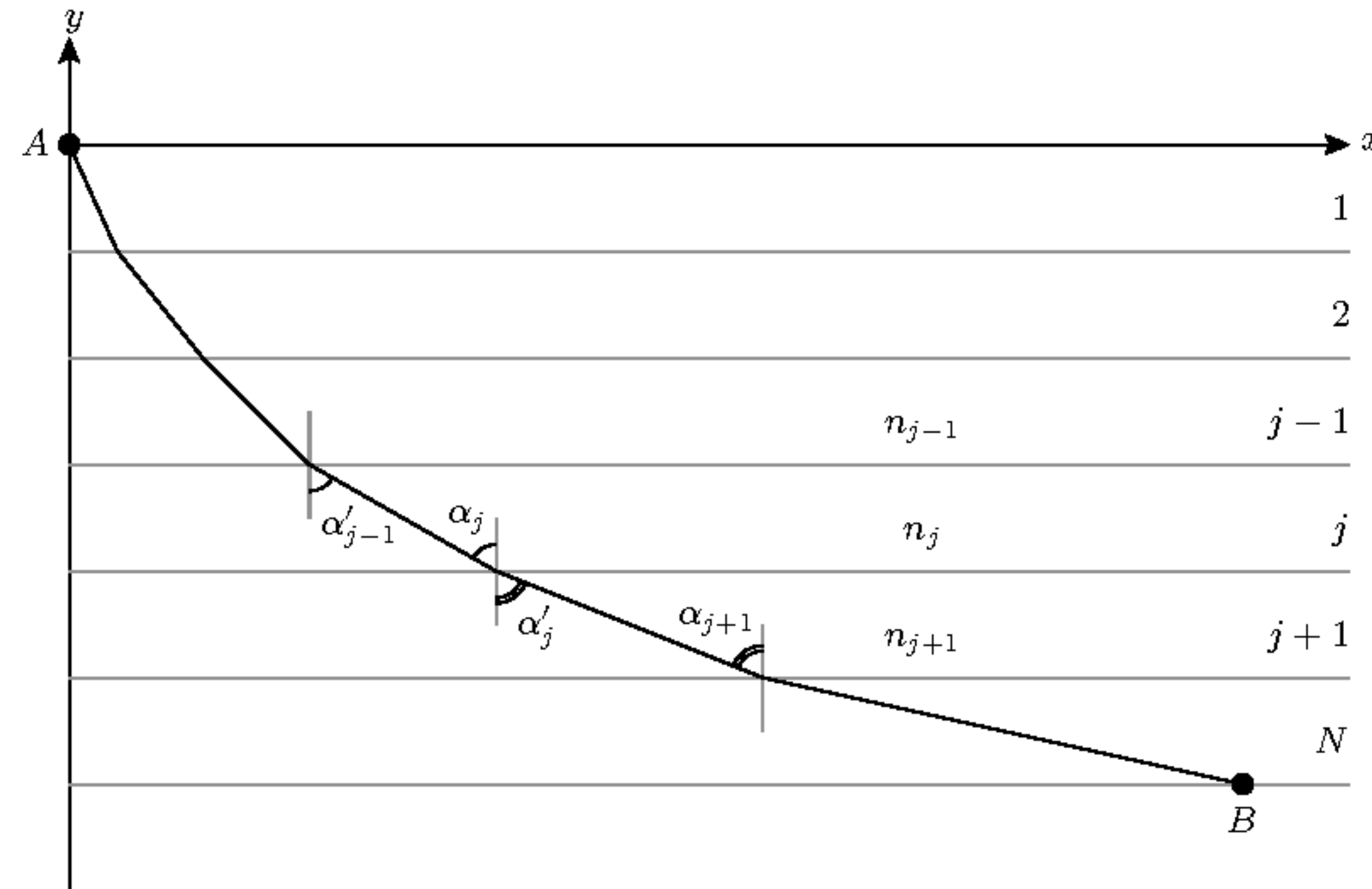
$$y = r(1 - \cos\theta)$$

$$x = \frac{1}{2\mathcal{C}}(\theta - \sin\theta)$$

$$y = \frac{1}{2\mathcal{C}}(1 - \cos\theta)$$

Bernoulli's approach to the brachistochrone problem

Idea: Light travels along the path that takes the shortest time.



$$\text{Snell's Law: } \frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2} = \dots = \frac{\sin \alpha_N}{v_N} \Rightarrow \frac{\sin \alpha}{v} = \text{constant}$$

Bernoulli's approach to the brachistochrone curve problem

Idea: Light travels along the path that takes the shortest time.

$$v = \sqrt{2gy}$$

$$\frac{\sin\alpha}{v} = \textit{constant}$$

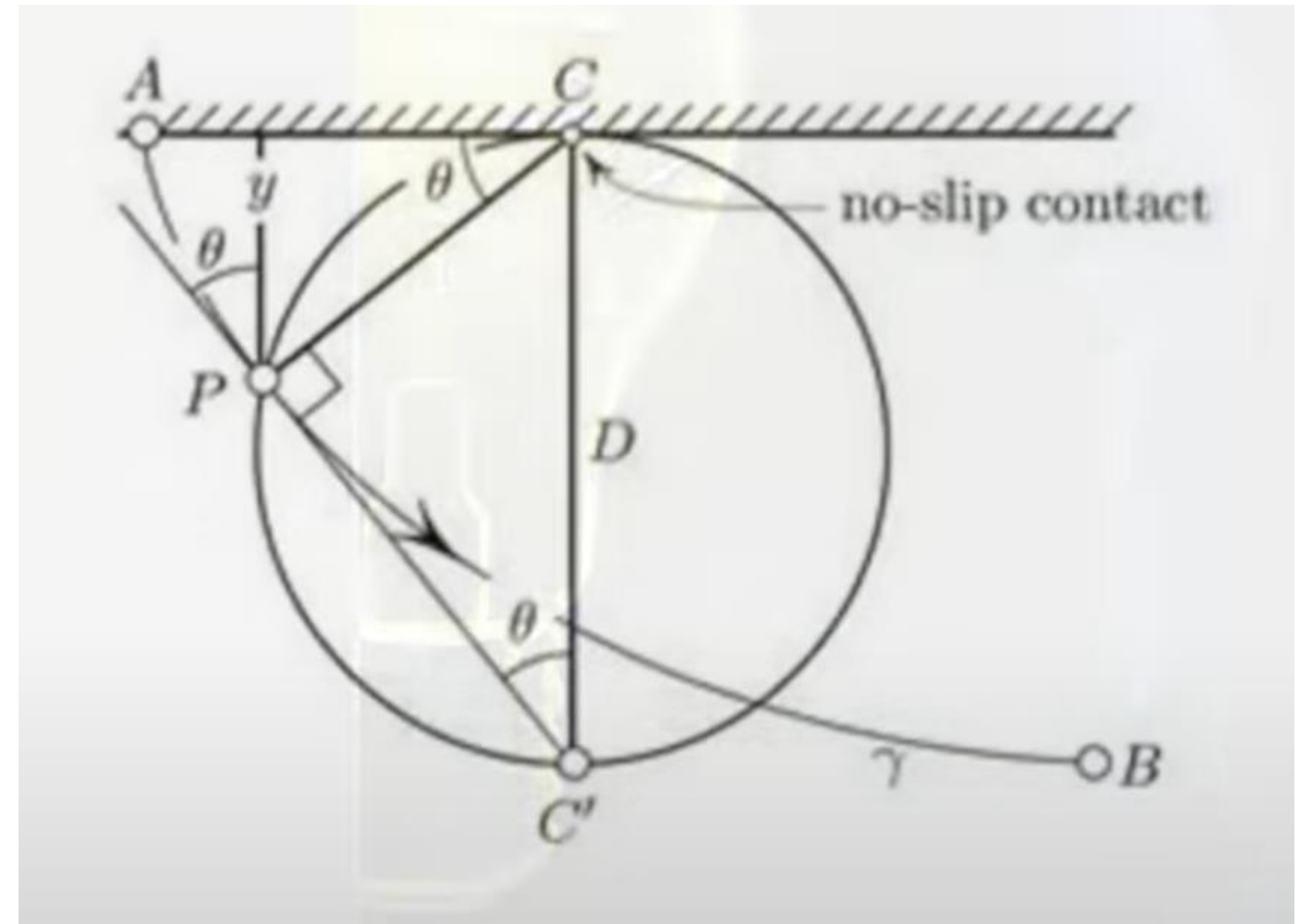
$$\frac{\sin\alpha}{\sqrt{y}} = \textit{constant}$$

Bernoulli's approach to the brachistochrone curve problem

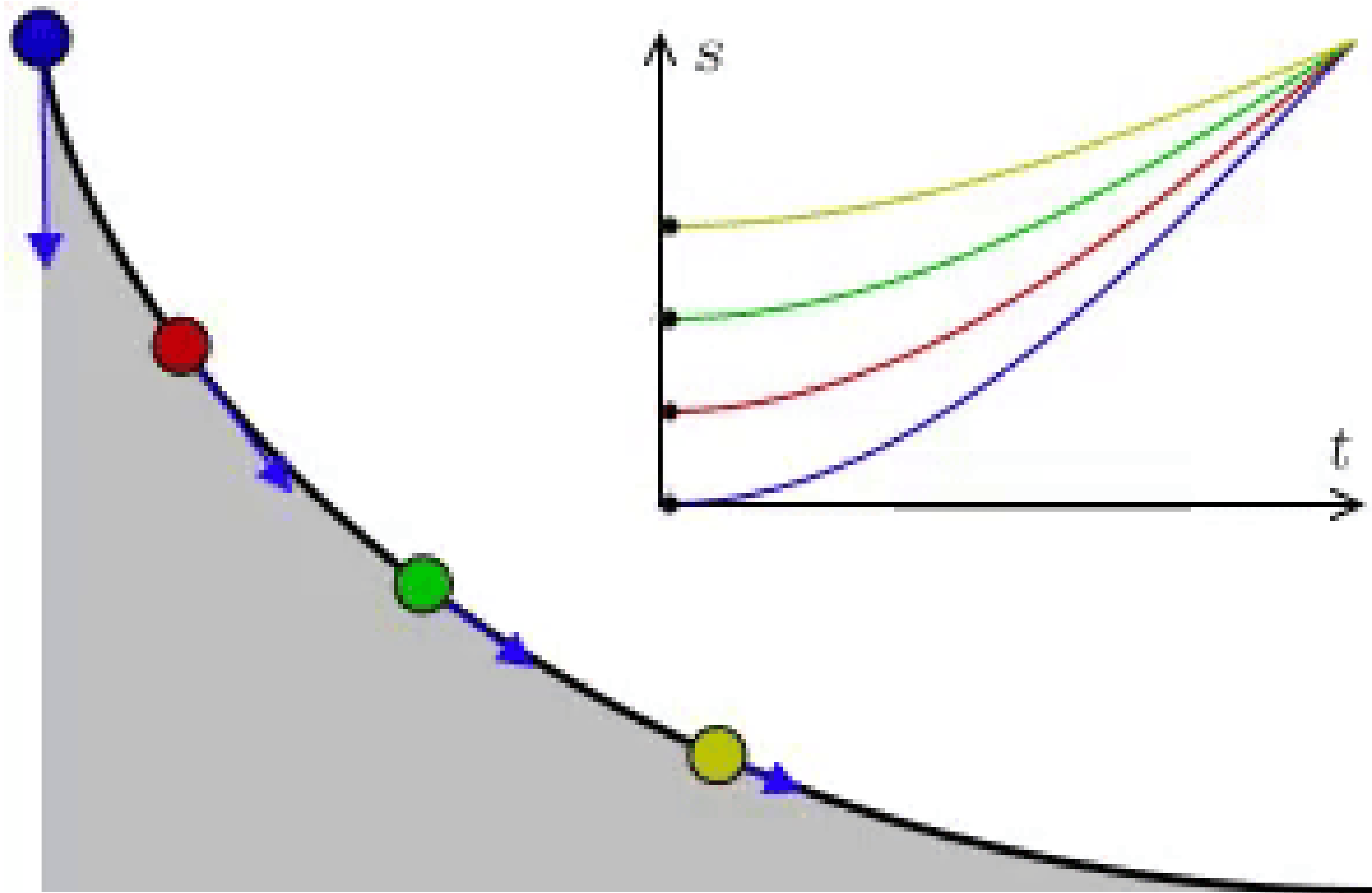
Idea: Light travels along the path that takes the shortest time.

$$y = \overline{CP} \sin \theta = [D \sin \theta] \sin \theta \\ = D \sin^2 \theta$$

$$\frac{\sin \theta}{\sqrt{y}} = \text{constant}$$



$$\Delta t = \text{constant}$$



$$\Delta t = \text{constant}$$

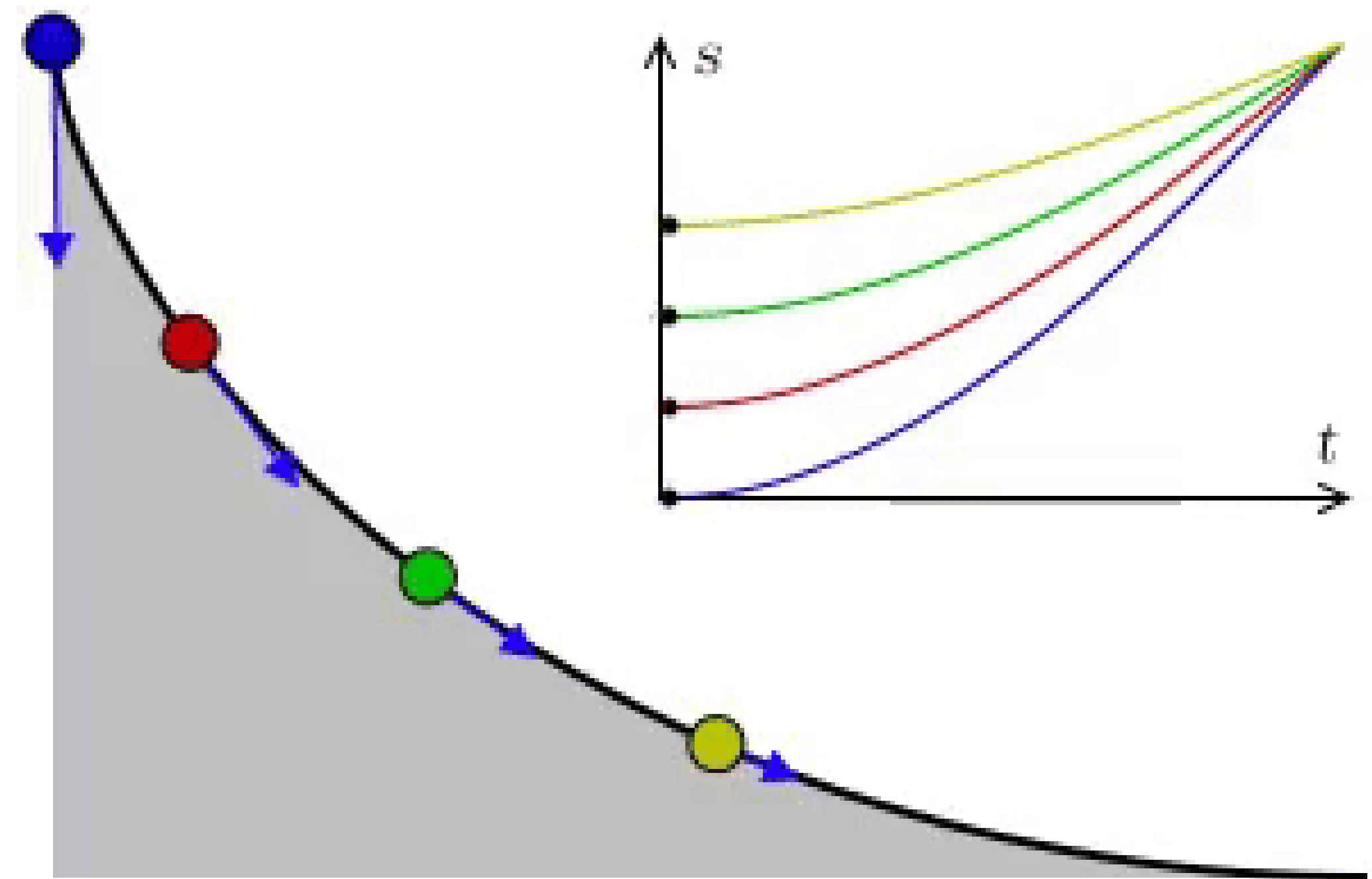
$$x = r(\theta - \sin\theta)$$

$$y = r(1 - \cos\theta)$$

$$0 = \Delta E_{\text{mech}} = \frac{1}{2}mv^2 + mg(y_i - y)$$

$$y_i = r(1 - \cos\theta_i), y = r(1 - \cos\theta)$$

$$v = \sqrt{2gr(\cos\theta_0 - \cos\theta)}$$



$$\Delta t = \text{constant}$$

$$dt = \frac{ds}{v} = \frac{ds}{\sqrt{2gr(\cos\theta_0 - \cos\theta)}}$$

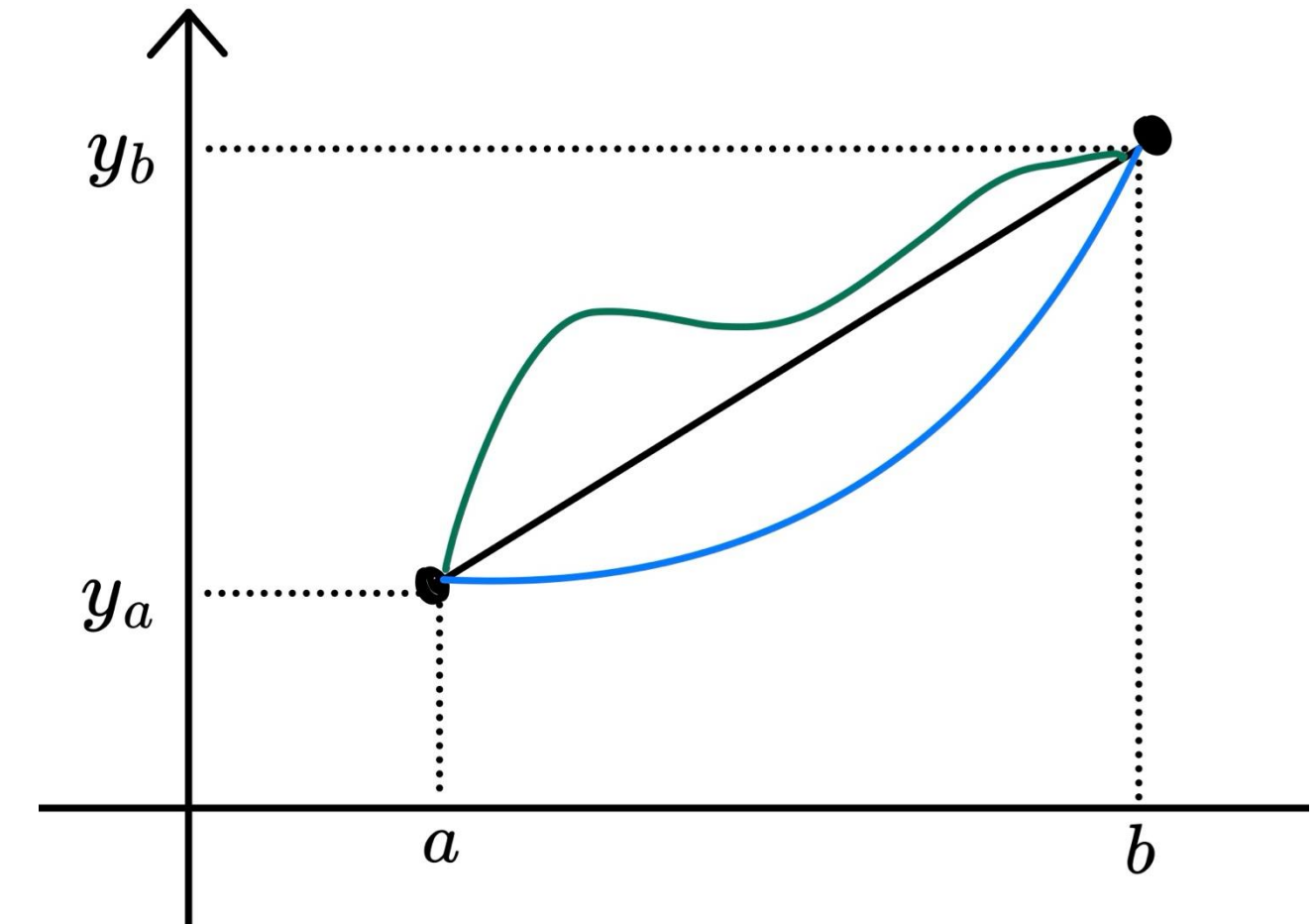
$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \sqrt{(r(1 - \cos\theta))^2 + (r\sin\theta)^2} d\theta \\ &= r\sqrt{2}\sqrt{1 - \cos\theta} d\theta = 2r\sin\frac{\theta}{2} d\theta \end{aligned}$$

$$\Delta t = \int_{\theta_i}^{\pi} \frac{2r\sin\frac{\theta}{2} d\theta}{\sqrt{2gr(\cos\theta_0 - \cos\theta)}} = \int_{\theta_i}^{\pi} \sqrt{\frac{r}{g}} \frac{\sin\frac{\theta}{2}}{\cos^2\frac{\theta_i}{2} - \cos^2\frac{\theta}{2}} d\theta = \pi \sqrt{\frac{r}{g}}$$

The shortest curve between two points

Goal: Minimize L

$$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



$$f = \sqrt{1 + (y')^2} = f(y, y') \Rightarrow \text{Use the Beltrami Identity}$$

$$f - y' \frac{\partial f}{\partial y'} = \text{constant} = C = \sqrt{1 + (y')^2} - \frac{(y')^2}{\sqrt{1 + (y')^2}} = \frac{1}{\sqrt{1 + (y')^2}}$$

The shortest curve between two points

Goal: Minimize L

$$\frac{1}{\sqrt{1+(y')^2}} = \text{constant} \Rightarrow y' = \text{constant} \Rightarrow y: \text{a straight line}$$