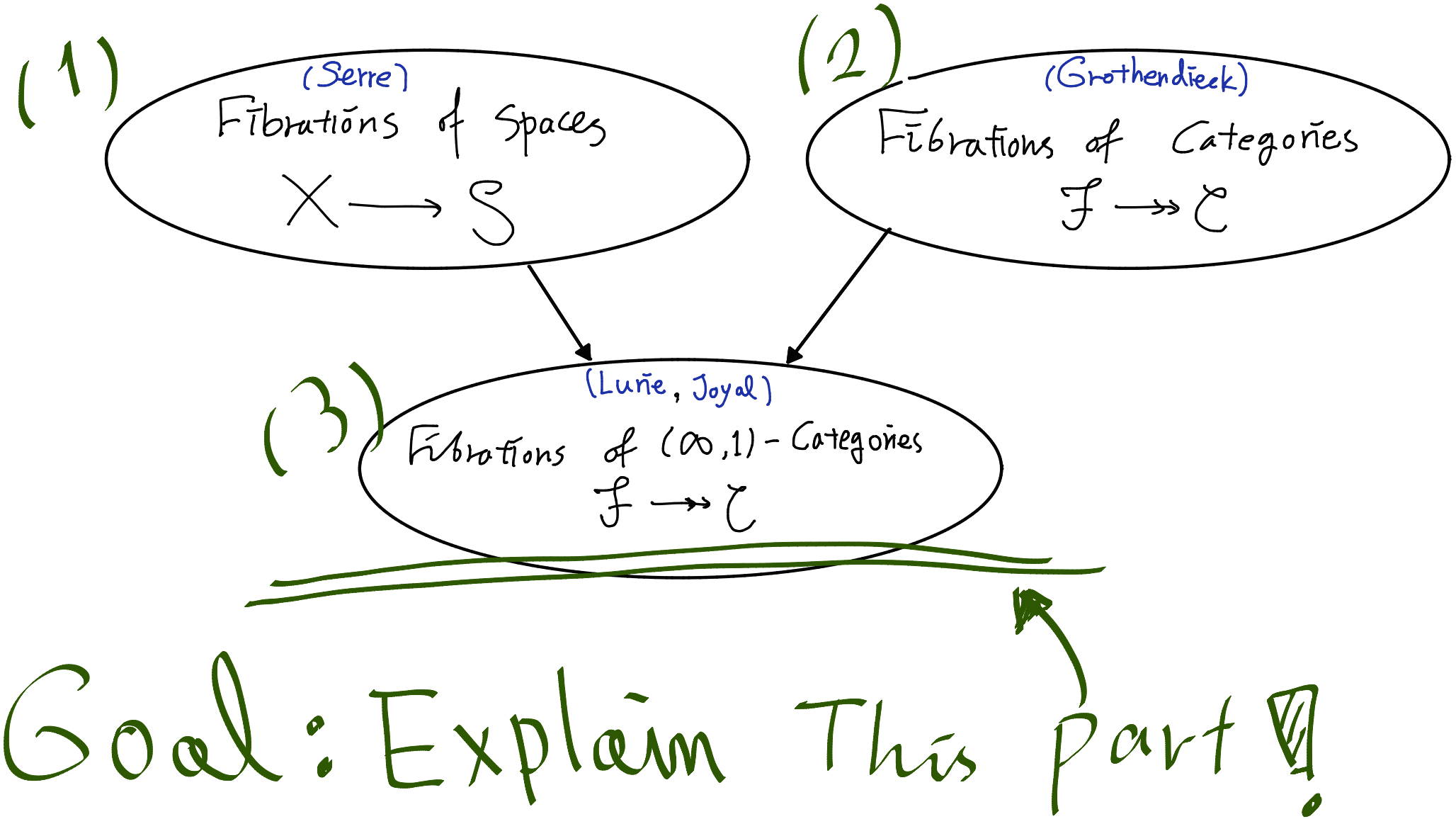
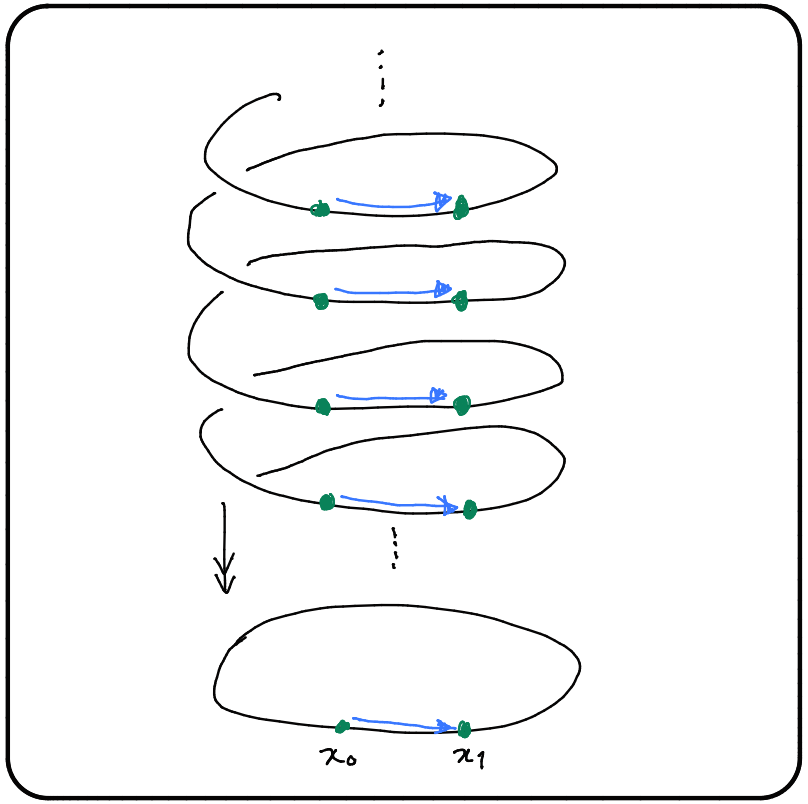
Fibrations in Modern Mathematics

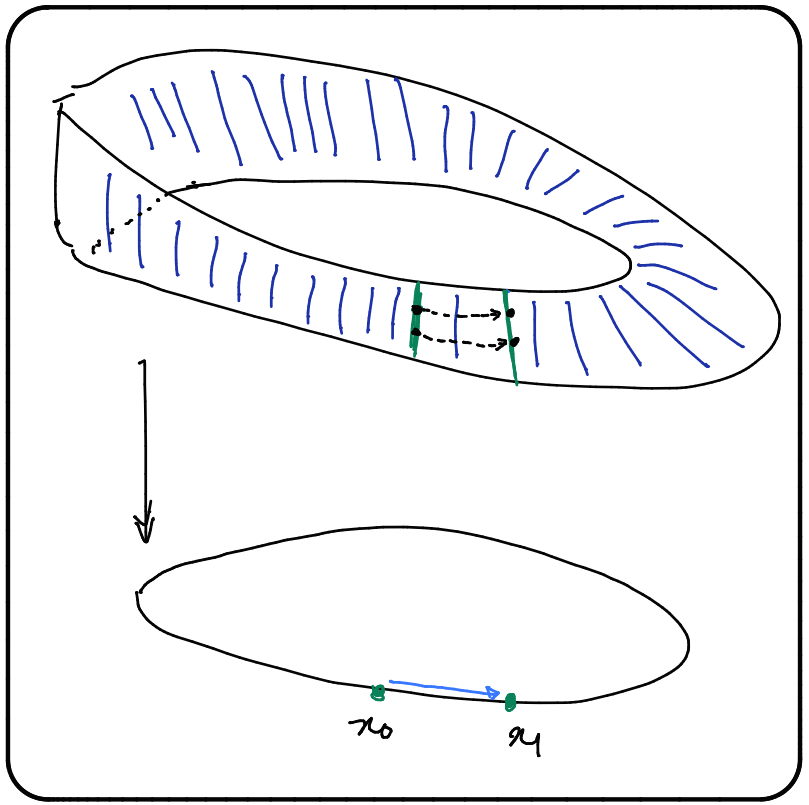


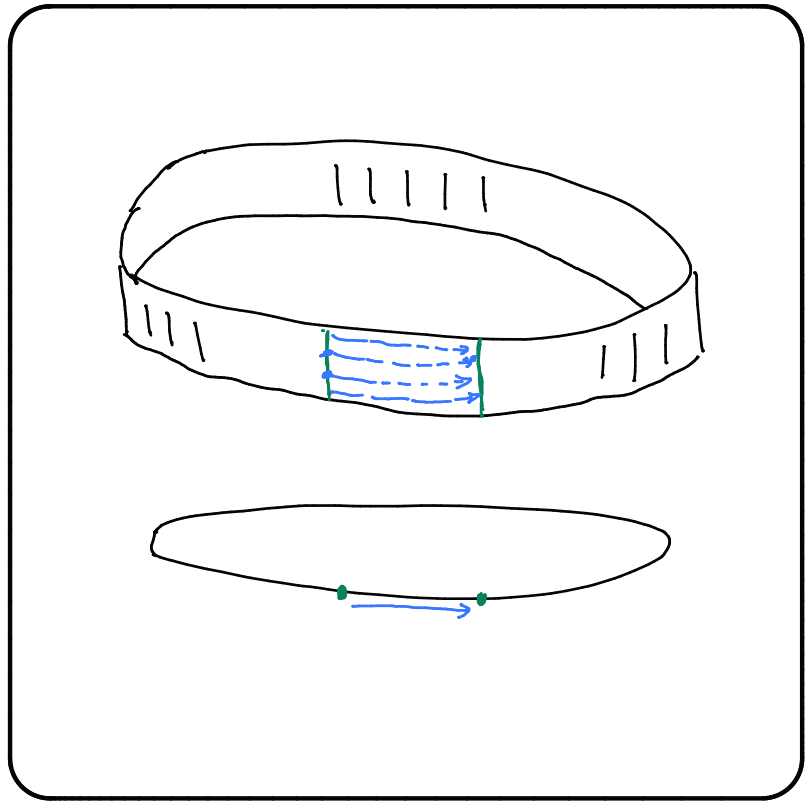
Goal: Explain This part!

(1) Fibrations of Spaces

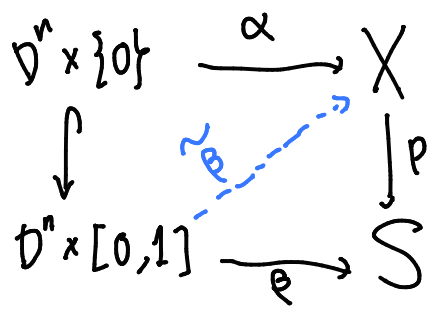
Examples



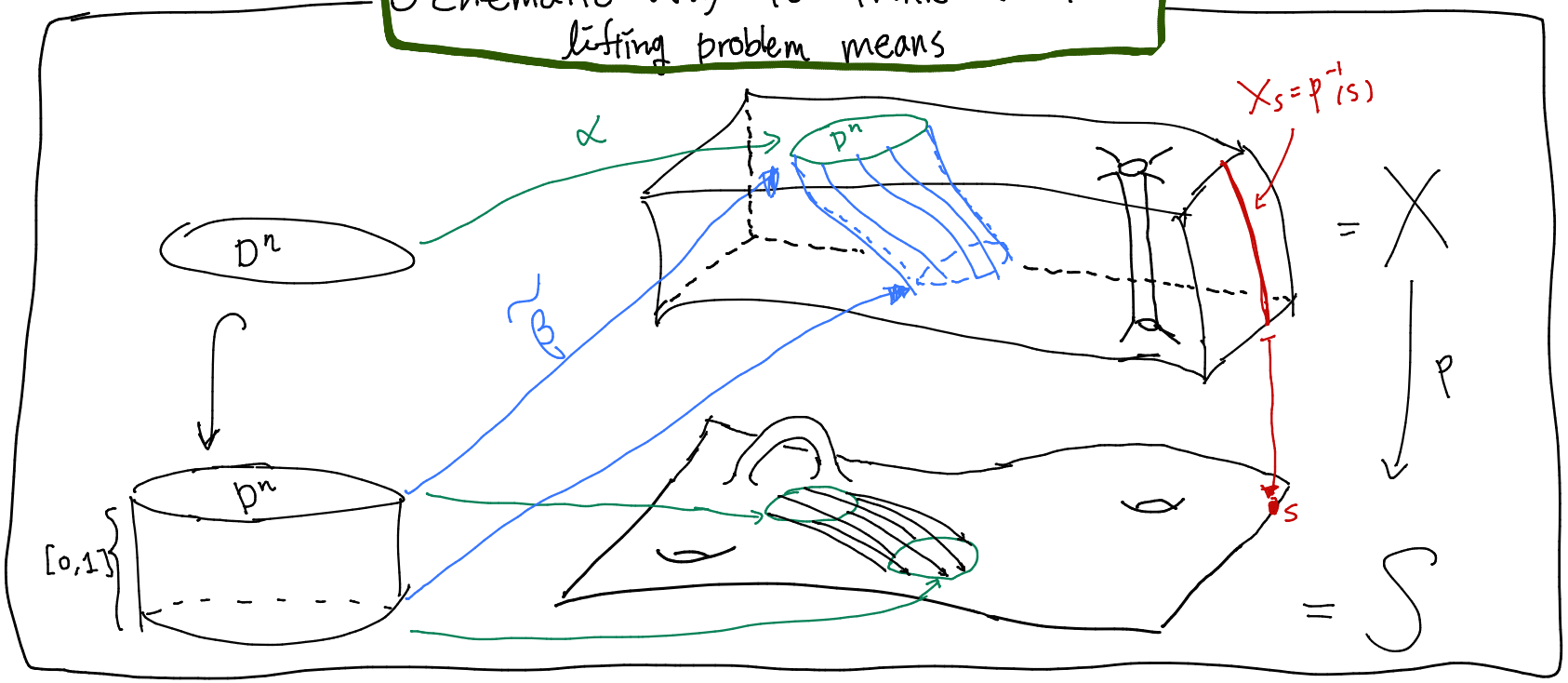




Def A map $X \xrightarrow{p} S$ between spaces is called fibration if for every comm. diagram as below, there exists $\tilde{\beta} : D^n \times [0,1] \to X$ s.t. $p \circ \tilde{\beta} = \beta$

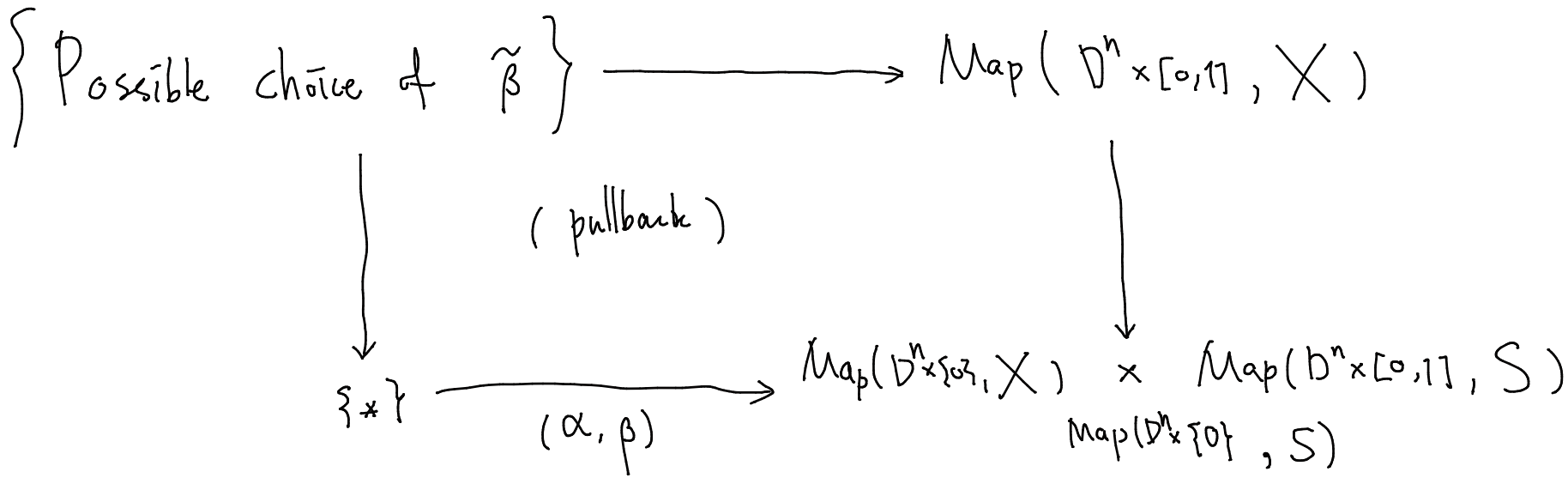


Schematic Way to think what lifting problem means

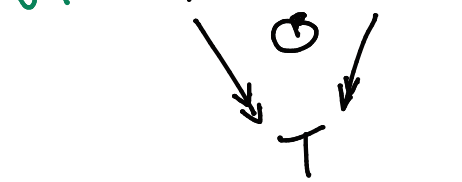


Prop. Any fiber bundle $E \twoheadrightarrow B$ is fibration

Remark. Surprisingly, there exists certain type of uniqueness of lifting:



Prop Given fibration $X \twoheadrightarrow S$ and maps $T \overset{f_0}{\underset{f_1}{\rightrightarrows}} S$ with homotopy $f_0 \simeq f_1$ (i.e. $H: T \times [0,1] \to S$ with $H(-,0) = f_0$, $H(-,1) = f_1$), we can find a map $g_H : f_0^* X \to f_1^* X$ of fibrations with homotopy $\left(f_0^* X \xrightarrow{g_H} f_1^* X \xrightarrow{pr_X} X \right) \simeq \left(f_0^* X \to X \right)$



Moreover, g_H is unique up to homotopy (i.e. any such g_H is homotopic to one another).

Informally, this suggests that

$$\left\{ \text{Homotopy } H: T \times [0,1] \to S \text{ from } f_0 \text{ to } f_1 \right\} \simeq \left\{ \text{Map from } f_0^* X \text{ to } f_1^* X \text{ over } T \right\}$$

proof. Pullback $X \xrightarrow{p} S$ along $H : T \times [0,1] \longrightarrow S$ and $T \times [0,1] \xrightarrow{pr_T} T \xrightarrow{f_0} S$

This represents constant homotopy from f_0 to f_0

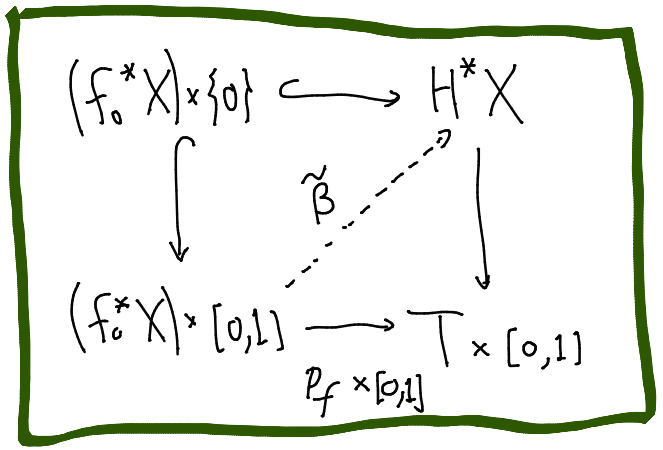
to get fibrations $\begin{pmatrix} H^*X \\ \downarrow \\ T \times [0,1] \end{pmatrix}$ and $\begin{pmatrix} (f_0^*X) \times [0,1] \\ \downarrow \\ T \times [0,1] \end{pmatrix}$ respectively and form a square as below

to get a solution $f_0^*X \times [0,1] \xrightarrow{\tilde{\beta}} H^*X$ over $T \times [0,1]$.

Pulling back this along $\iota_1 : T \times \{1\} \hookrightarrow T \times [0,1]$, we finally get a map $f_0^*X \longrightarrow f_1^*X$.

$$\begin{array}{ccc} f_0^*X & & f_1^*X \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

Others follow similarly. ∎



We usually apply this to $T = \{pt\}$, then the homotopy H just means path between two points s_0, s_1 and $g_H : X_{s_0} \to X_{s_1}$ is exactly the parallel transport "type" map, called homotopy transport of H

Observations

(1) Given two paths $\gamma, \delta : [0,1] \to S$ with $\gamma(1) = \delta(0)$ we define $(\delta * \gamma) : [0,1] \to S$ by $t \mapsto \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \delta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$. Then there exists homotopy between map



$$g_\delta \circ g_\gamma \simeq g_{(\delta \circ \gamma)} \quad (\star)$$

(2) From this property, we can easily deduce that any g_γ must be an homotopy equivalence.

To summary what I've said, we shall go into a Category theory little

Def A Category $\mathcal{C}$ consists of following datum with conditions :

(1) A collection $Ob(\mathcal{C})$ of objects. We'll say X is object of $\mathcal{C}$ if $X \in Ob(\mathcal{C})$

(2) Sets $Hom_{\mathcal{C}}(X,Y)$ for each object $X, Y \in \mathcal{C}$, called morphism from X to Y

We'll write $f : X \longrightarrow Y$ if $f \in Hom_{\mathcal{C}}(X,Y)$

(3) For each $X, Y, Z \in \mathcal{C}$ we have composition

$$Hom_{\mathcal{C}}(Y,Z) \times Hom_{\mathcal{C}}(X,Y) \xrightarrow{\circ} Hom_{\mathcal{C}}(X,Y) \quad \text{by } (g,f) \longmapsto g \circ f$$

Satisfying (a) identity , (b) associativity

Examples

(1) Category $\underline{\mathbf{Set}}$ with

$$Ob(Set) = \{ \text{all sets} \}$$

$$\mathrm{Hom}_{Set}(X,Y) = \{ \text{all functions } X \xrightarrow{f} Y \}$$

(2) Category $\underline{\mathbf{Vect}_{\mathbb{R}}}$ with

$$Ob(\mathrm{Vect}_{\mathbb{R}}) = \{ \text{Vector spaces} \}$$

$$\mathrm{Hom}_{\mathrm{vect}_{\mathbb{R}}}(V,W) = \{ \mathbb{R}\text{-linear map } V \xrightarrow{f} W \}$$

(3) Category $\underline{\mathbf{Top}}$ with

$$Ob(Top) = \{ \text{all topological spaces} \}$$

$$\mathrm{Hom}_{Top}(X,Y) = \{ \text{continuous map } X \longrightarrow Y \}$$

(4) Category $\underline{h_0(\mathbf{Top})}$ with

$$Ob(h_0(Top)) = \{ \text{all topological spaces} \}$$

$$\mathrm{Hom}_{h_0Top}(X,Y) = \{ \text{continuous map } X \longrightarrow Y \} / \text{homotopy}$$

Every algebraic structures (group, ring, vector space) has homomorphism between them

and so does Category

Def A Functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ between categories consists of following datum:

(1) A function $Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$ by $X \longmapsto F(X)$

(2) For each $X, Y \in \mathcal{C}$, functions $Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X), F(Y))$

$$f \longmapsto F(f)$$

Satisfies some compatibility

$$\begin{cases} (a) \quad F(id_X) = id_{F(X)} \\ (b) \quad F(g \circ f) = F(g) \circ F(f) \end{cases}$$

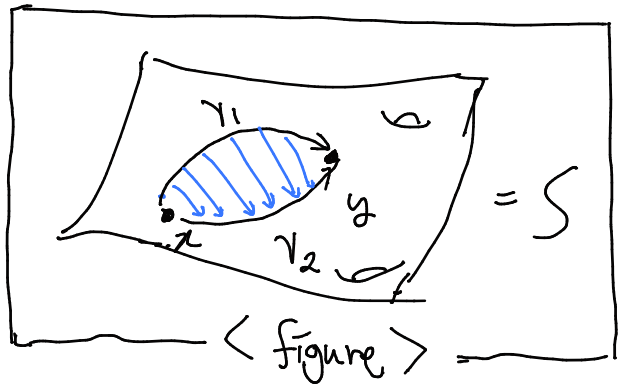
Now another definition arises.

Def Let S be a space. The fundamental groupoid $\pi_{\leq 1}(S)$ of a Category with

(1) $Ob(\pi_{\leq 1} S) = S$ as set. i.e. $\{\text{Objects of } \pi_{\leq 1} S\} = \{\text{points of } S\}$

(2) $\mathrm{Hom}_{\pi_{\leq 1} S}(x, y) = \{\text{Path } \gamma : [0,1] \longrightarrow S \text{ with } \gamma(0)=x,\ \gamma(1)=y\} /$ homotopy relation $\simeq$ (relative to x, y)

$\gamma_1 \simeq \gamma_2 \iff$ There exists continuous deformation like figure



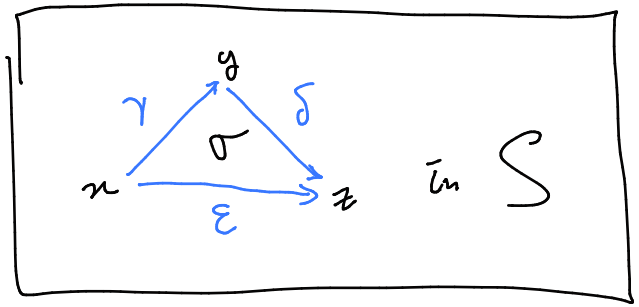
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Remark Giving a functor $\mathcal{L} : \Pi_{\leq 1} S \to \mathcal{C}$ is same as giving following datum:

(1) For any point $x \in S$, an object $\mathcal{L}_x \in \mathcal{C}$

(2) For any path γ from x to y, a map $\gamma_! : \mathcal{L}_x \to \mathcal{L}_y$ in $\mathcal{C}$

(3) For 2-simplex $\Delta^2 \xrightarrow{\sigma} S$, we have $\varepsilon_! = \delta_! \circ \gamma_! : \mathcal{L}_x \to \mathcal{L}_z$



Def Let $\mathrm{Fib}(S)$ be a category that objects are fibration $X \xrightarrow{p} S$ and morphism betwen $X \xrightarrow{p} S$ and $Y \xrightarrow{q} S$ is commuting square $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$

Construction

We have a following functor $\underline{St}$, called straightening functor

$$St : \mathrm{Fib}(S) \longrightarrow \mathrm{Fun}(\pi_{\leq 1}(S), h_0(Top))$$

$$(p : X \to S) \longmapsto \left[\left\{ \begin{array}{ccccc} & & y & & \\ & \gamma \nearrow & & \searrow \delta & \\ x & & \xrightarrow{\varepsilon} & & z \end{array} \text{ in } S \right\} \mapsto \left\{ \begin{array}{ccc} & g_\gamma & p^{-1}(y) \\ p^{-1}(x) & & \\ & \xrightarrow{g_\varepsilon} & p^{-1}(z) \end{array} \right\} \right]$$



Question Can we recover fibration $X \longrightarrow S$ from $\pi_{\leq 1}(S) \longrightarrow h_0(\mathrm{Top})$?

Partially Yes ! but no , in general . (or should say terribly no.)

Thm (Fundamental theorem of covering space) Roughly speaking , a covering space $X \xrightarrow{p} S$ is fibration with discrete fibers. Then we have an equivalence of categories

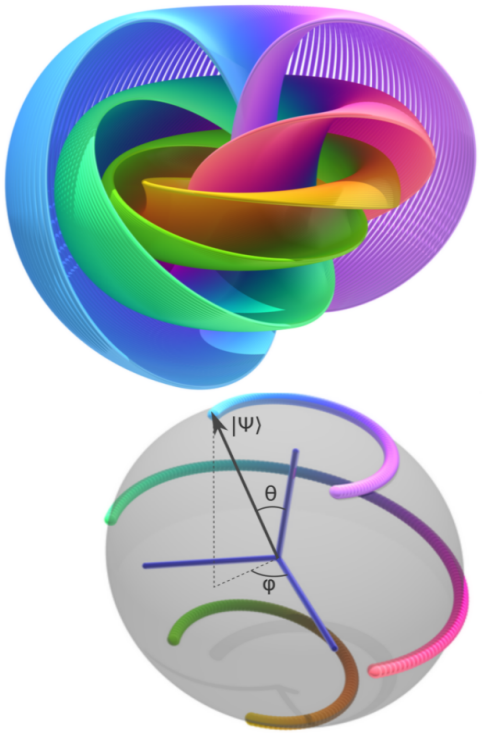
$$St : Cov(S) \xrightarrow{\sim} Fun(\pi_{\leq 1} S, Set)$$

by restricting St to covering spaces

Example First, note that $\pi_{\leq 1}(S^2) \simeq \{\text{Trivial category}\}$. So $\mathrm{Fun}(\pi_{\leq 1} S^2, ho(\mathrm{Top})) \cong ho(\mathrm{Top})$

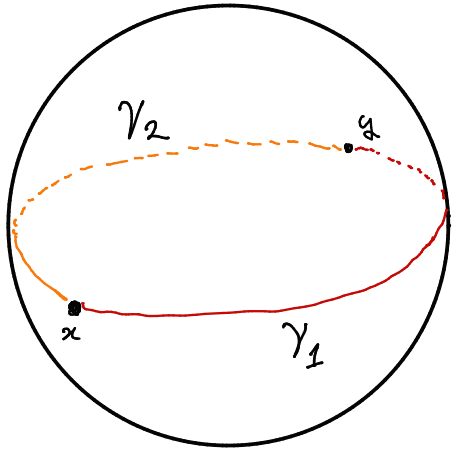
This says, any $\pi_{\leq 1} S^2 \longrightarrow ho(\mathrm{Top})$ detects only the trivial fibration $pr_{S^2} : S^2 \times F \to S^2$

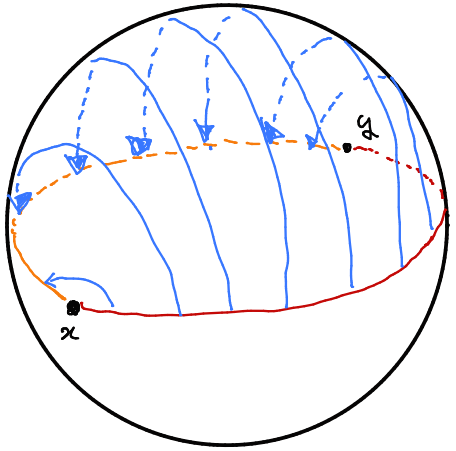
On the other hand, we have non-trivial fibration $S^3 \to S^2$ with fiber S^1. (Hopf's)



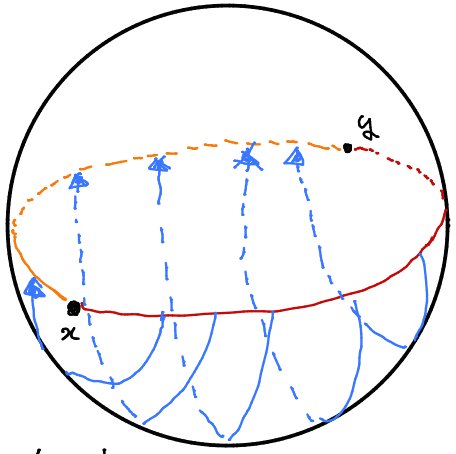
< Hopf fibration >

Here's some explaination. Consider two paths γ_1, γ_2 through equator





⟨homotopy $N : \gamma_1 \simeq \gamma_2$⟩



⟨homotopy $S : \gamma_1 \simeq \gamma_2$⟩

Two are not homotopic to one another relative to equator $\cong S^1$.

But $\pi_{\leq 1}(S^2)$ is too truncated to capture the difference between N and S

This is the cause of all problem.

So if we want a theorem of the form $Fib(S) \longrightarrow Fun(\pi_? S, Spaces)$

(1) We have to enhance the notion of fundamental groupoid $\Pi_{\leq 1}(-)$ to contain whole higher informations of spaces (homotopy, homotopy between homotopy, homotopy between homotopy between homotopy, etc ...)

(2) To do so, we first have to enhance the notion of category so that above has meaning, also should replace $ho(Top)$ by Category of Spaces, here Category in the sense of such enhancement

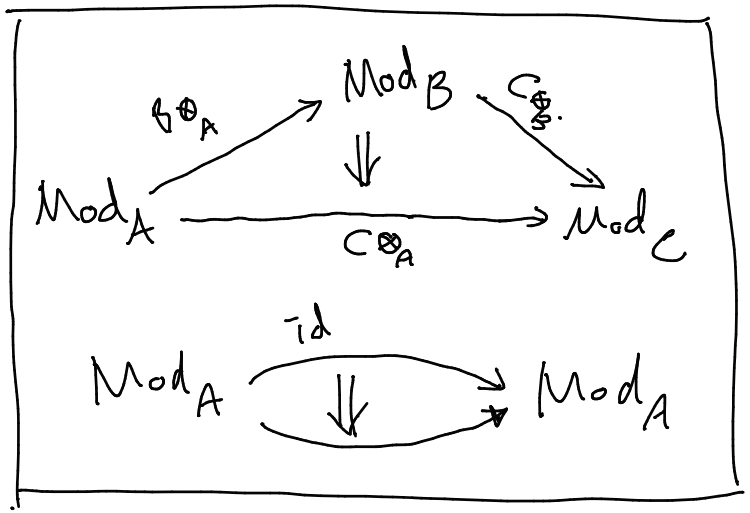
This kind of observation leads us to a notion of $(\infty, 1)$-Category.

(2) Fibration of Categories.

From now on, let me assume that the audiences are familiar with Category theory

Motivation

Assignment $\mathrm{CRing} \longrightarrow \mathrm{Cat}$ by $A \longmapsto \mathrm{Mod}_A$ and $(A \xrightarrow{f} B) \longmapsto (\mathrm{Mod}_A \xrightarrow{B \otimes_A (-)} \mathrm{Mod}_B)$ isn't a functor



but $\exists\ C \otimes_B (B \otimes_A M) \xrightarrow{\sim} C \otimes_A M$

$\exists\ M \xrightarrow{\sim} A \otimes_A M$

(+ Suitable compatibility)

Def A pseudofunctor $F : \mathcal{C} \longrightarrow Cat$ from $\mathcal{C}$ to Cat consists of following datum

(1) A function $Ob(\mathcal{C}) \longrightarrow Ob(Cat)$

(2) For each $X, Y \in \mathcal{C}$, a functor $\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{Cat}(F(X), F(Y))$

with only identities, i.e. discrete category

(3) For each $X \in \mathcal{C}$, a nat. isom. $\varepsilon_X : id_{F(X)} \xRightarrow{\sim} F(id_X)$

(4) For each $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, a nat. isom. $\mu_{g,f} : F(g) \circ F(f) \xRightarrow{\sim} F(g \circ f)$

Moreover for $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$, we have commutative diagram of nat. isom.

$$\begin{array}{ccc} F(h) \circ F(g) \circ F(f) & \xRightarrow{\mu_{h,g}} & F(h \circ g) \circ F(f) \\ \Big\Downarrow{\scriptstyle \mu_{g,f}} & & \Big\Downarrow{\scriptstyle \mu_{h \circ g, f}} \\ F(h) \circ F(g \circ f) & \xRightarrow[\mu_{h, g \circ f}]{} & F(h \circ g \circ f) \end{array}$$

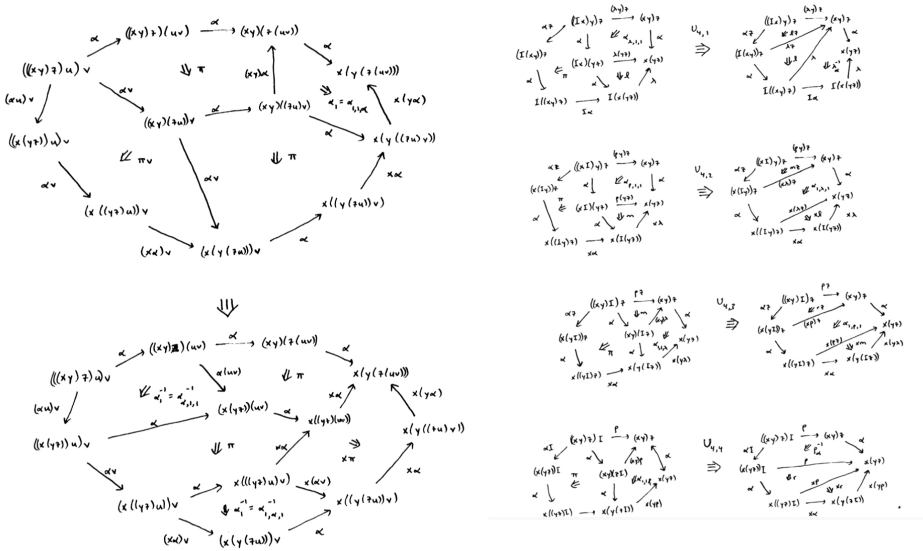
Example

(1) $\mathrm{Mod}_{(-)} : \mathrm{CRing} \longrightarrow \mathrm{Cat}$ is pseudofunctor

(2) $\mathrm{Vect}(-) : \mathrm{Top}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ is pseudofunctor.

Remark

You must believe this way of defining higherly coherent datum will get harder and harder very quikly.



(Some coherence conditions for 3-category)

OR, we just form a fibration of category $?\twoheadrightarrow \mathcal{C}$ by

Construction

Define a category $\int_{\mathcal{C}} F$ as follows

$$Ob(\int_{\mathcal{C}} F) = \{ (a, X) : X \in F(a) \text{ with } c \in \mathcal{C} \}$$

A morphism from (a, X) to (b, Y) is
① $a \xrightarrow{f} b$ in $\mathcal{C}$
② $F(f)(X) \xrightarrow{\alpha} Y$ in $F(b)$

Then we have projection $\int_{\mathcal{C}} F \longrightarrow \mathcal{C}$ by $(a, X) \longmapsto a$

This assembles all datum contained in $\mathcal{C} \xrightarrow{F} Cat$

Example

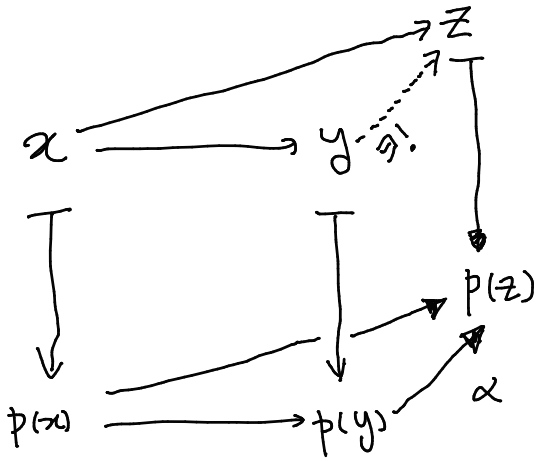
$$\int_{\mathrm{CRing}} \mathrm{Mod}_{(-)} = \{ (A, M) : A \in \mathrm{CRing} \text{ and } A\text{-module } M \}$$

$$(f, \alpha) : (A, M) \longrightarrow (B, N) \iff f : A \to B \text{ and } B \otimes_A M \xrightarrow{\alpha} N$$

with $p : \int_{\mathrm{CRing}} \mathrm{Mod}_{(-)} \longrightarrow \mathrm{CRing}$. Note that $p^{-1}(A) = \mathrm{Mod}_A$

What's the fibration of a Cat?

Def Let $E \xrightarrow{p} \mathcal{C}$ be a functor and let $f: x \to y$ be a map in E. Then $f: x \to y$ is said to be p-coCartesian if for every $x \xrightarrow{g} z$ and comm. $p(x) \xrightarrow{p(g)} p(z)$, $p(x) \xrightarrow{p(f)} p(y) \xrightarrow{\alpha} p(z)$ there exists unique lifting $\tilde{\alpha}$ of α. Or equivalently we have pullback square



$$\begin{array}{ccc} \mathrm{Hom}_E(y,z) & \xrightarrow{\circ f} & \mathrm{Hom}_E(x,z) \\ \downarrow p & & \downarrow p \\ \mathrm{Hom}_{\mathcal{C}}(p(y),p(z)) & \xrightarrow[\circ p(f)]{} & \mathrm{Hom}_{\mathcal{C}}(p(x),p(z)) \end{array} \quad (\forall z \in E)$$

Example In $\int_{CRing} Mod_{(-)}$, $(A,M) \xrightarrow{(f,\alpha)} (B,N)$ is p-coCartesian if and only if $B \otimes_A M \xrightarrow{\alpha} N$ is an isomorphism.

Def $E \xrightarrow{p} \mathcal{C}$ is said to be coCartesian fibration if for any map $f: x \to y$ in $\mathcal{C}$ with $p(\tilde{x}) = x$, there exists p-coCartesian lift $\tilde{x} \longrightarrow f_! \tilde{x}$ of $f: x \to y$

By choosing $f_!(\tilde{x})$ for every $\tilde{x} \in p^{-1}(x) = E_x$, we can actually form a functor between fibers: $E_x \xrightarrow{f_!} E_y$ by $\tilde{x} \longmapsto f_!(\tilde{x})$.

But because of occuring of the choice, the construction does not garantee

$$g_! \circ f_! = (g \circ f)_! \qquad (\text{for } x \xrightarrow{f} y \xrightarrow{g} z \text{ in } \mathcal{C})$$

But we still have $g_! \circ f_! \xRightarrow{\sim} (g \circ f)_!$ and $id_{E_x} \xRightarrow{\sim} id_{x!}$ (+ other compatibility)

Example In a case of $\int_{\mathrm{CRing}} \mathrm{Mod} \longrightarrow \mathrm{CRing}$, for any $A \xrightarrow{f} B$ and $(A, M) \in \int_{\mathrm{CRing}} \mathrm{Mod}$ has p-coCartesian lift, namely, $(A, M) \longrightarrow (B, B \otimes_A M)$

So $\left(\int_{\mathrm{CRing}} \mathrm{Mod}\right)_A = \mathrm{Mod}_A \xrightarrow{f_!} \mathrm{Mod}_B = \left(\int_{\mathrm{CRing}} \mathrm{Mod}\right)_B$ is exactly the $B \otimes_A (-)$.

And this is not a coincidence:

Thm (Grothendieck Construction) We have

$$\mathrm{St} : \mathrm{coCart}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{pseudo}}(\mathcal{C}, \mathrm{Cat})$$

$$(E \twoheadrightarrow \mathcal{C}) \longmapsto \left[\left\{ x \xrightarrow{f} y \xrightarrow{g} z \right\} \mapsto \left\{ E_x \xrightarrow{f_!} E_y \xrightarrow{g_!} E_z \text{ with } E_x \xrightarrow{(fog)_!} E_z \right\} \right]$$

with inverse $\int_{\mathcal{C}} (-)$

Def A coCartesian fibration is said to be left fibration if every map in E is p-coCartesian

Def A coCartesian fibration is said to be discrete fibration if p-coCartesian lift $\tilde{\gamma}$ is unique

Thm (Grothendieck Construction) There exist equivalences of categories

$$\begin{array}{ccc}
\mathrm{coCart}(\mathcal{C}) & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{pseudo}}(\mathcal{C}, \mathrm{Cat}) \\
\uparrow & & \uparrow \\
\mathrm{LFib}(\mathcal{C}) & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{pseud.}}(\mathcal{C}, \mathrm{Gpd}) \\
\uparrow & & \uparrow \\
\mathrm{Dis\ Fib}(\mathcal{C}) & \xrightarrow{\sim} & \mathrm{Fun}(\mathcal{C}, \mathrm{Set})
\end{array}$$

$$\left\{\begin{array}{l}\text{Study of pseudofunctors} \\ \text{i.e. functor + (little of higher coherence)}\end{array}\right\} \simeq \left\{\text{Study of fibrations } E \xrightarrow{} \mathcal{C} \xleftarrow{} F\right\}$$

This suggests that it may be a good idea to keep dealing with fibrations, because of the difficulty of other side.

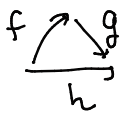
(3) Fibrations of ∞-categories.

Def$^{(1)}$ A ∞-category $\mathcal{C}_\bullet$ consists of following datum

(0) $\mathcal{C}_0 = \{$objects of $\mathcal{C}\}$

dom ↑ ↓ id ↑ cod

(1) $\mathcal{C}_1 = \{$morphisms in $\mathcal{C}\}$



g ↑ ↓ ↑ h ↓ ↑ f

(2) $\mathcal{C}_2 = \{$homotopies $\alpha : h \simeq g \circ f$ for each composible maps g, f in $\mathcal{C}\}$

↑ ↓ ↑ ↓ ↑ ↓

(3) $\mathcal{C}_3 = \{$homotopies between homotopies of maps in $\mathcal{C}\}$

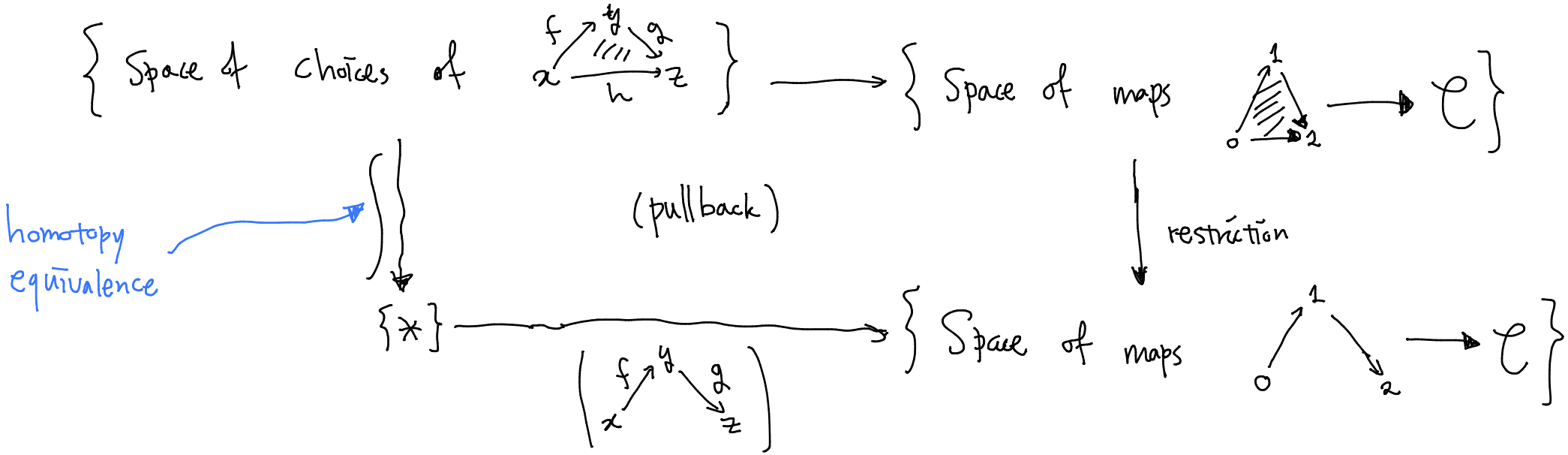
↑ ↓ ↑ ↓ ↑ ↓ ↑

⋮

with a certain lifting property

Def (2) / Thm A ∞-category $\mathcal{C}$ is category where $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is Spaces and the composition of $x \xrightarrow{f} y \xrightarrow{g} z$ can only defined up to contractible choice

i.e.



As in usual category theory, we can define Cat_∞, a ∞-category of ∞-categories

Def We say ∞-category S is ∞-groupoid if every morphism in S is invertible

Set $Gpd_\infty \subseteq Cat_\infty$ be a full subcategory spaned by ∞-groupoids.

Thm (Grothendieck's Homotopy hypothesis)

We have equivalence of ∞-categories $Top = Spaces \xrightarrow[\pi_{\leq \infty}]{\sim} Gpd_\infty$

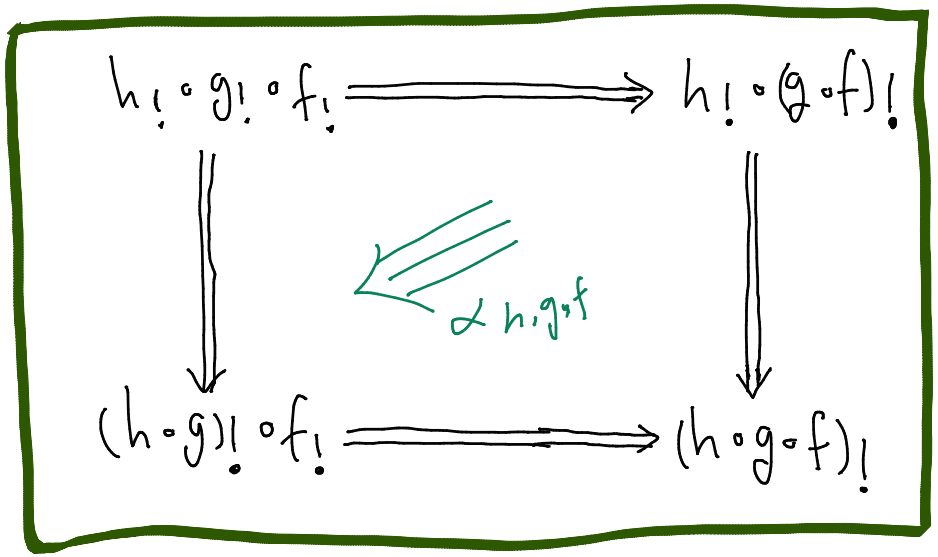
Remark We have ∞-categorical counterparts of the notions of fibrations of a category you just have to change the part "pullback" to homotopy pullback and

"unique" to unique up to contractible choice

Construction

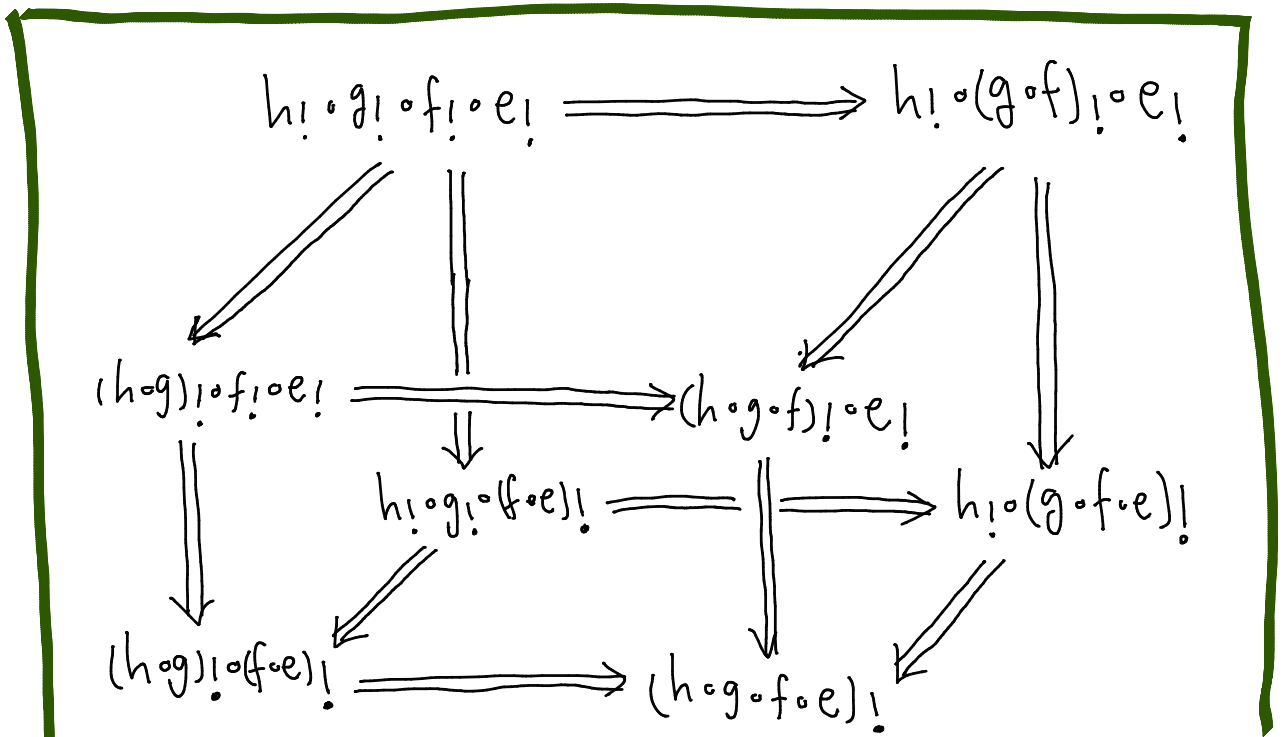
We can mimic previous construction for coCart. fibration $E \twoheadrightarrow \mathcal{C}$

But it is not sufficient to remember $g_! \circ f_! \xRightarrow{\sim} (g \circ f)_!$. We have to remember



, name this to $\alpha_{h,g,f}$.

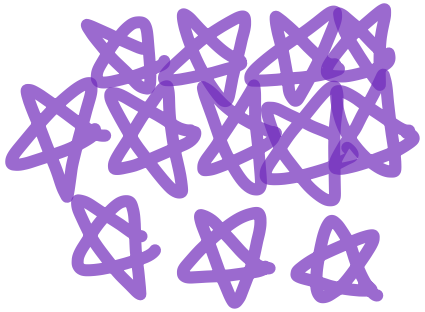
Then we again have to remember



The whole cube defines $\beta_{h,g,f,e}$

and so on -----

Thm (∞-Grothendieck Construction, by Lurie)



We have equivalences of categories for ∞-category $\mathcal{C}$

$$\begin{array}{ccc} \mathrm{St}: \mathrm{coCart}(\mathcal{C}) & \xrightarrow{\sim} & \infty\text{-}\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \\ \uparrow & & \uparrow \\ \mathrm{St}: \mathrm{LFib}(\mathcal{C}) & \xrightarrow{\sim} & \infty\text{-}\mathrm{Fun}(\mathcal{C}, \mathrm{Gpd}_\infty) \end{array}$$

Moreover, when $\mathcal{C}$ is an ∞-groupoid, i.e. of the form $\tau_{\leq\infty}(S)$, we have

$$\mathrm{Fib}(S) := \mathrm{LFib}(\tau_{\leq\infty}[S]) \xrightarrow{\sim} \infty\text{-}\mathrm{Fun}(\tau_{\leq\infty} S, \mathrm{Gpd}_\infty)$$

This answers to (1)

Thank you!

Thm (∞-Yoneda lemma) Let $\mathcal{C}$ be an ∞-category

Let $h_{\mathcal{C}} : \mathcal{C} \longrightarrow \infty\text{-}\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Gpd}_\infty) = \hat{\mathcal{C}}$ be yoneda lemma.

Then we have functorial $\mathrm{Hom}_{\hat{\mathcal{C}}}(h_{\mathcal{C}}(x), F) \simeq F(x)$

<u>proof Strategy</u> Both assignment has own left fibrations $\int_{\mathcal{C}^{op} \times \hat{\mathcal{C}}} \mathrm{Hom}_{\hat{\mathcal{C}}}(h_{\mathcal{C}}[-], -)$ and $\int_{\mathcal{C}^{op} \times \hat{\mathcal{C}}} ev$

Then we can find zig-zag of an isom.

$$\int_{\mathcal{C}^{op} \times \hat{\mathcal{C}}} \mathrm{Hom}_{\hat{\mathcal{C}}}(h_{\mathcal{C}}[-], [-]) \xleftarrow{\sim} ? \xrightarrow{\sim} \int_{\mathcal{C}^{op} \times \hat{\mathcal{C}}} ev$$

$$\mathcal{C}^{op} \times \hat{\mathcal{C}}$$