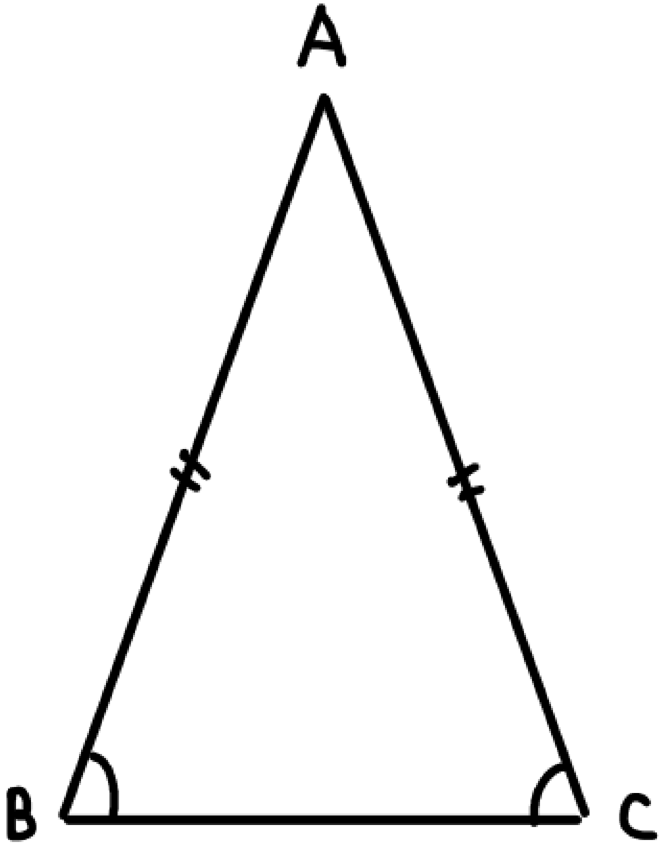
Shortest Proofs in Mathematics

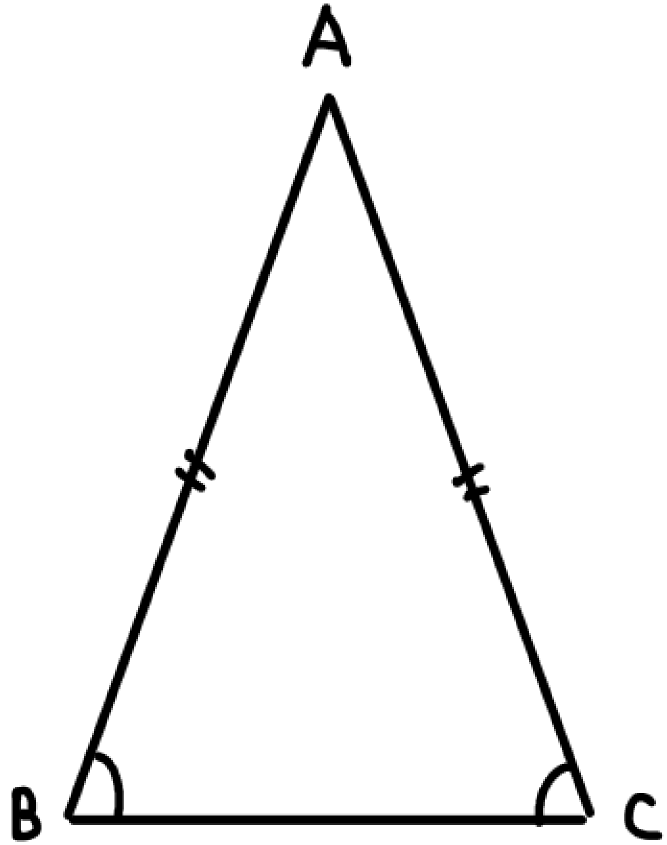
김종근

들어가기에 앞서...



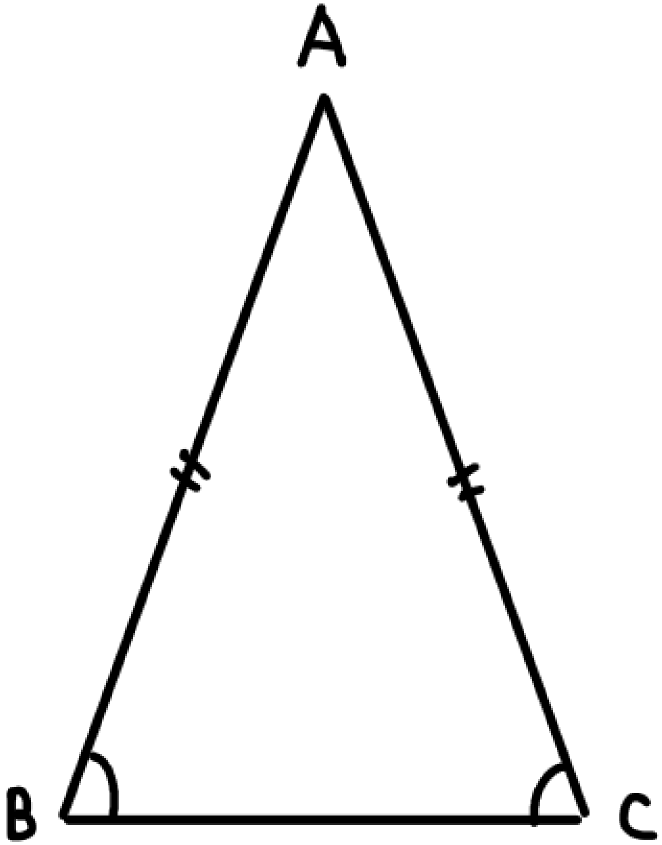
$AB = AC$인 삼각형 $\triangle ABC$

들어가기에 앞서…



삼각형 $\triangle ABC$ 에서 각B와 각C 가 같음을 보이시오.

들어가기에 앞서...



$\triangle BAC$ 와 $\triangle CAB$ 는 SAS 합동, 따라서 $\angle B = \angle C.\ Q.E.D.$

1. 페르마의 마지막 정리
(Fermat’s last Theorem)

에 대한 오일러의 추측

페르마의 마지막 추측

$n > 2$ 인 정수일 때, $x^n + y^n = z^n$ 을 만족하는 0이 아닌 정수해 (x, y, z)가 존재하지 않는다.

페르마의 마지막 추측

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=> 1995년 Andrew Wiles가 증명 완료!

오일러의 추측

좌변의 항이 3개 이상인 경우에는 n = 3 일 때 해가 존재

$$ex)\ 3^3 + 4^3 + 5^3 = 6^3$$

=> 지수가 n 이라면, 좌변의 항이 최소 n 개 필요하지 않을까?

오일러의 거듭제곱 합 추측

$a_1^n + a_2^n + \ \dots + a_k^n = b^n$ 을 만족하는 자연수 a_i, b 가 존재하려면 $k \geq n$ 여야 한다.

=> 페르마의 마지막 정리는 $\mathrm{k} = 2$ 인 특수한 경우

오일러의 거듭제곱 합 추측

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1769년에 제안, 그 후 약 200년 동안 증명되지 않았음

COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [**1**] that at least n nth powers are required to sum to an nth power, $n > 2$.

Reference

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REFERENCE

또 다른 반례

1988년 Noam Elkies가 $n = 4, k = 3$ 인 경우 다음과 같은

무한개의 반례가 있다는 것을 증명

$$(85v^2 + 484v - 313)^4 + (68v^2 - 586v + 10)^4 + (2u)^4 = (357v^2 - 204v + 363)^4,$$

where

$$u^2 = 22030 + 28849v - 56158v^2 + 36941v^3 - 31790v^4.$$

또 다른 반례

$n \geq 6$ 인 경우 아직 정수해가 존재하는지 미해결

($k = n$ 인 경우 포함)

2. 페르마의 두 제곱수 정리
(Fermat's Theorem on sums of two squares)

홀수 소수의 패턴 관찰

3, 5, 7, 11, 13, 17, 19, 23, 27, 31, 35, …

홀수 소수의 패턴 관찰

3, 5, 7, 11, 13, 17, 19, 23, 27, 31, …

알베르 지라르(Albert Girard)

홀수 소수의 패턴 관찰

3, 5, 7, 11, 13, 17, 19, 23, 27, 31, …

흠....

알베르 지라르(Albert Girard)

홀수 소수의 패턴 관찰

$$3, 5, 7, 11, 13, 17, 19, 23, 27, 31, \ldots$$

$\big\downarrow$ 4로 나누었을 때 나머지

홀수 소수의 패턴 관찰

3, 5, 7, 11, 13, 17, 19, 23, 27, 31, …

4로 나누었을 때 나머지

3, 1, 3, 3 , 1 , 1 , 3 , 3 , 3 , 3 , …

홀수 소수의 패턴 관찰

3, 5, 7, 11, 13, 17, 19, 23, 27, 31, 35, …

4로 나누었을 때 나머지

3, 1, 3, 3 , 1 , 1 , 3 , 3 , 3 , 3 , 3 , …

나머지가 1인 소수들을 관찰

나머지가 1인 소수의 패턴 관찰

$$5 = 1^2 + 2^2$$

$$13 = 2^2 + 3^2$$

$$17 = 1^2 + 4^2$$

$$\vdots$$

두 제곱수의 합으로 나타낼 수 있음을 1625년
알베르 지라르(Albert Girard)가 처음으로 문헌에 남김

나머지가 1인 소수의 패턴 관찰

$$5 = 1^2 + 2^2$$

$$13 = 2^2 + 3^2$$

$$17 = 1^2 + 4^2$$

$$\vdots$$

피에르 드 페르마 (Pierre de Fermat)

나머지가 1인 소수의 패턴 관찰

$$5 = 1^2 + 2^2$$

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$$17 = 1^2 + 4^2$$

$$\vdots$$

쉽네 ㅋㅋ

그 후 페르마는 마랭 메르센(Marin Mersenne)에게 “4로 나누어 1이 남는 모든 소수는 제곱수의 합으로 나타낼 수 있다”고 확언

나머지가 1인 소수의 패턴 관찰

페르마는 자신이 고안한 무한 강하법(Method of Infinite Descent)를 통해 정리를 증명했다 주장, 하지만 편지에 "증명 과정이 너무 길어서 생략한다"고만 적어두고 명확한 풀이를 남기지 않았음

아 성님...

더 이상의 자세한 설명은 생략한다.

마랭 메르센 (Marin Mersenne)

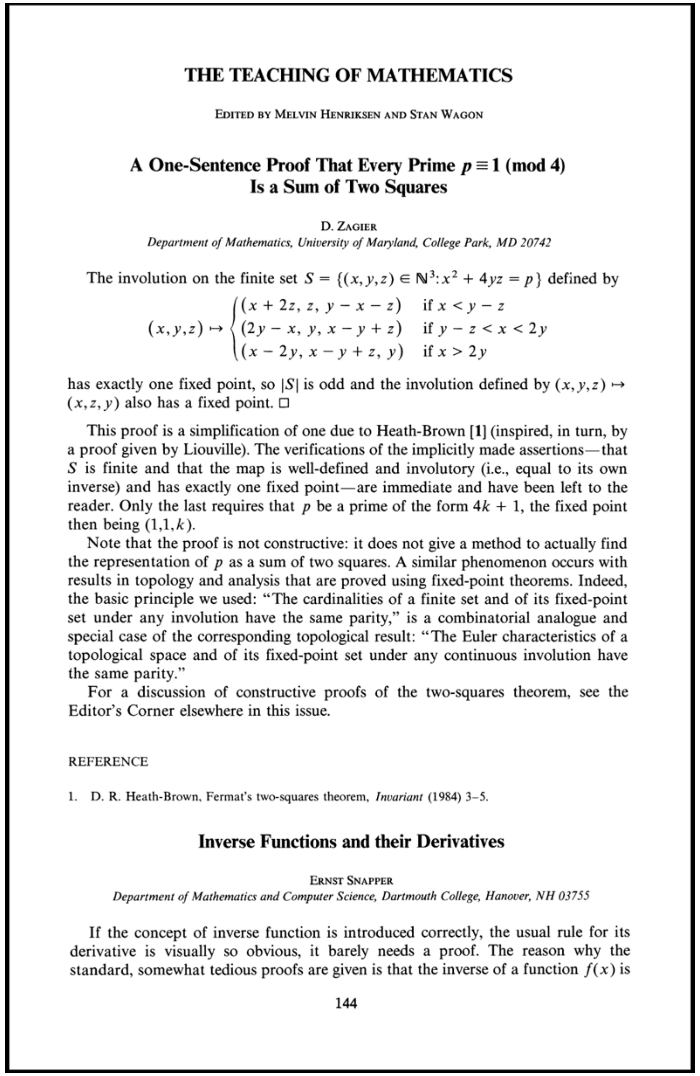
나머지가 1인 소수의 패턴 관찰

이후, 오일러가 이 정리를 증명하는 데 무려

7년(1742~1749년)을 쏟아부었음

하......

레온하르트 파울 오일러(Leonhard Paul Euler)



EDITED BY MELVIN HENRIKSEN AND STAN WAGON

A One-Sentence Proof That Every Prime $p \equiv 1 \pmod 4$ Is a Sum of Two Squares

D. ZAGIER
Department of Mathematics, University of Maryland, College Park, MD 20742

The involution on the finite set $S = \{(x,y,z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$ defined by

$$(x,y,z) \mapsto \begin{cases} (x+2z,\ z,\ y-x-z) & \text{if } x < y-z \\ (2y-x,\ y,\ x-y+z) & \text{if } y-z < x < 2y \\ (x-2y,\ x-y+z,\ y) & \text{if } x > 2y \end{cases}$$

has exactly one fixed point, so $|S|$ is odd and the involution defined by $(x,y,z) \mapsto (x,z,y)$ also has a fixed point. □

This proof is a simplification of one due to Heath-Brown [**1**] (inspired, in turn, by a proof given by Liouville). The verifications of the implicitly made assertions—that S is finite and that the map is well-defined and involutory (i.e., equal to its own inverse) and has exactly one fixed point—are immediate and have been left to the reader. Only the last requires that p be a prime of the form $4k+1$, the fixed point then being $(1,1,k)$.

Note that the proof is not constructive: it does not give a method to actually find the representation of p as a sum of two squares. A similar phenomenon occurs with results in topology and analysis that are proved using fixed-point theorems. Indeed, the basic principle we used: "The cardinalities of a finite set and of its fixed-point set under any involution have the same parity," is a combinatorial analogue and special case of the corresponding topological result: "The Euler characteristics of a topological space and of its fixed-point set under any continuous involution have the same parity."

For a discussion of constructive proofs of the two-squares theorem, see the Editor's Corner elsewhere in this issue.

REFERENCE

1. D. R. Heath-Brown, Fermat's two-squares theorem, *Invariant* (1984) 3–5.

Inverse Functions and their Derivatives

ERNST SNAPPER
Department of Mathematics and Computer Science, Dartmouth College, Hanover, NH 03755

If the concept of inverse function is introduced correctly, the usual rule for its derivative is visually so obvious, it barely needs a proof. The reason why the standard, somewhat tedious proofs are given is that the inverse of a function $f(x)$ is

1990년, 돈 자기에(Don Zagier)의 논문이 등장

THE TEACHING OF MATHEMATICS

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오일러가 7년간 고민했던 문제를 단 한 페이지로 해결

Zagier의 풀이법

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

유한 집합 S를 생각

Zagier의 풀이법

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

S에서 S로 가는 함수 f를 정의

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이 함수는 특징이 있음

Zagier의 풀이법

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두 번 합성하면 항등 함수가 됨

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두 번 합성하면 항등 함수가 됨

$$(x,y,z) \to f(x,y,z) \to f\big(f(x,y,z)\big) = (x,y,z)$$

Zagier의 풀이법

$$(x, y, z) \mapsto \begin{cases} (x + 2z,\ z,\ y - x - z) & \text{if } x < y - z \\ (2y - x,\ y,\ x - y + z) & \text{if } y - z < x < 2y \\ (x - 2y,\ x - y + z,\ y) & \text{if } x > 2y \end{cases}$$

함수 f의 성질을 통해 집합 S에서 짝을 지을 수 있음.

$$(x, y, z) \Leftrightarrow f(x, y, z)$$

Zagier의 풀이법

$$(x,y,z) \mapsto \begin{cases} (x+2z,\ z,\ y-x-z) & \text{if } x < y-z \\ (2y-x,\ y,\ x-y+z) & \text{if } y-z < x < 2y \\ (x-2y,\ x-y+z,\ y) & \text{if } x > 2y \end{cases}$$

하지만 함숫값이 변하지 않는 고정점이라면, 짝을 지을 수 없음

$$(x,y,z) = f(x,y,z)?$$

Zagier의 풀이법

$$(x, y, z) \mapsto \begin{cases} (x + 2z,\ z,\ y - x - z) & \text{if } x < y - z \\ (2y - x,\ y,\ x - y + z) & \text{if } y - z < x < 2y \\ (x - 2y,\ x - y + z,\ y) & \text{if } x > 2y \end{cases}$$

$(x, y, z) = f(x, y, z)$이 되는 경우는 $(1,1,\frac{p-1}{4})$ 밖에 존재하지 않음

Zagier의 풀이법

$$(x, y, z) \mapsto \begin{cases} (x + 2z,\ z,\ y - x - z) & \text{if } x < y - z \\ (2y - x,\ y,\ x - y + z) & \text{if } y - z < x < 2y \\ (x - 2y,\ x - y + z,\ y) & \text{if } x > 2y \end{cases}$$

고정점이 단 하나이므로, 집합 S의 원소의 개수는 홀수

$$(x, y, z) \Leftrightarrow f(x, y, z), \qquad (1,1,\frac{p-1}{4})$$

Zagier의 풀이법

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

S에서 S로 가는 함수 g를 정의,

$$g(x, y, z) = (x, z, y)$$

Zagier의 풀이법

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

S에서 S로 가는 함수 g를 정의,

$$\mathrm{g}(x, y, z) = (x, z, y)$$

이 함수 또한 두 번 합성하면 항등 함수가 됨

Zagier의 풀이법

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

$$\mathrm{g}(x, y, z) = (x, z, y)$$

따라서 g를 통해 S의 원소들을 짝지을 수 있고,
홀수 개이므로 고정점이 반드시 하나 존재

$(x, y, z) \Leftrightarrow g(x, y, z),$ $\qquad$ (? , ? , ?)

Zagier의 풀이법

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

$$(x, y, z) = g(x, y, z) = (x, z, y)$$

$\Rightarrow y = z$ 인 순서쌍 존재

Zagier의 풀이법

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

$$(x, y, z) = g(x, y, z) = (x, z, y)$$

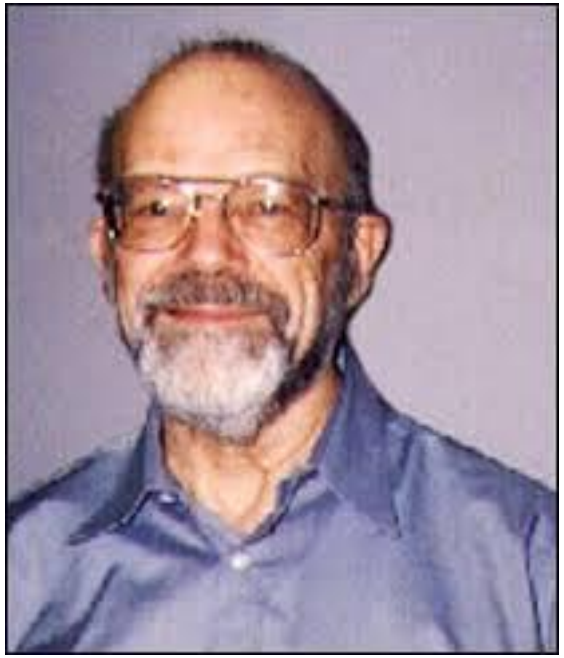
$\Rightarrow y = z$ 인 순서쌍 존재

$$\therefore x^2 + 4yz = x^2 + 4y^2 = x^2 + (2y)^2 = p$$

를 만족시키는 자연수 x, y가 존재

3. 미술관 정리
(Chvátal’s Art Gallery Theorem)

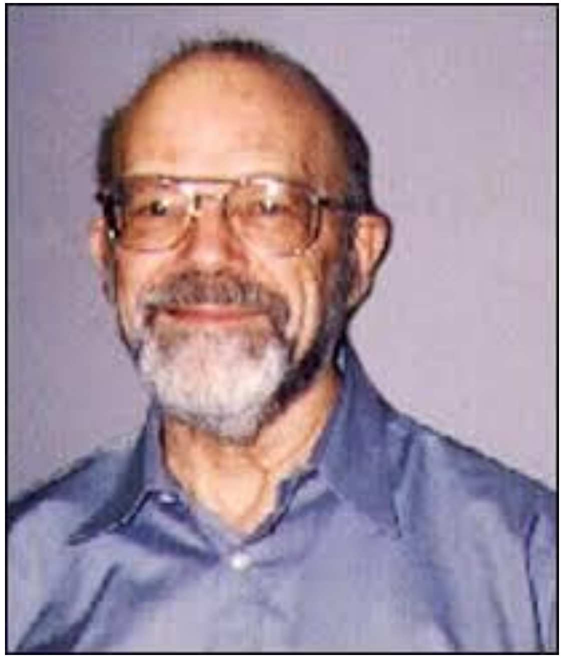
1973년 스탠퍼드 대학교에서 열린 기하학 학회에서 시작

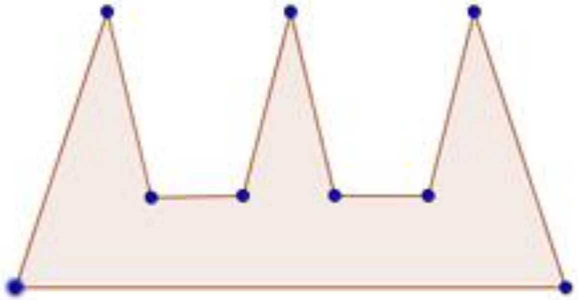


빅터 클리(Victor Klee)

바츨라프 흐바탈(Václav Chvátal)

다각형 모양의 미술관 전체를
사각지대 없이 감시하려면,
꼭짓점 개수 n개에 따라 경비원이
최대 몇 명이나 필요할까요?





다각형 모양의 미술관 전체를
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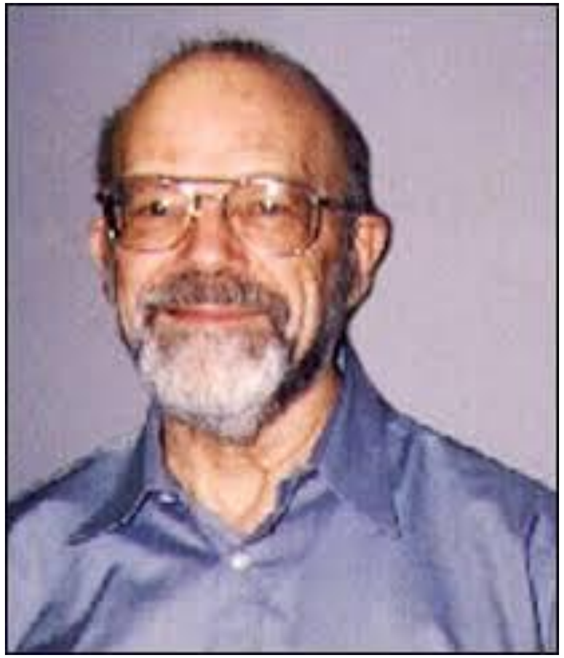
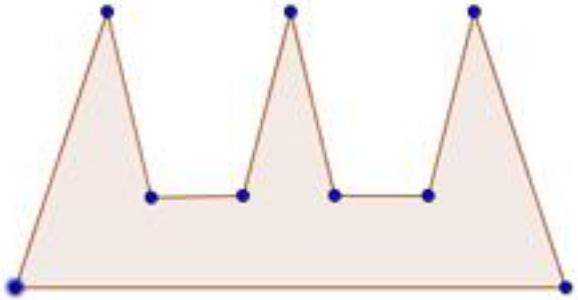
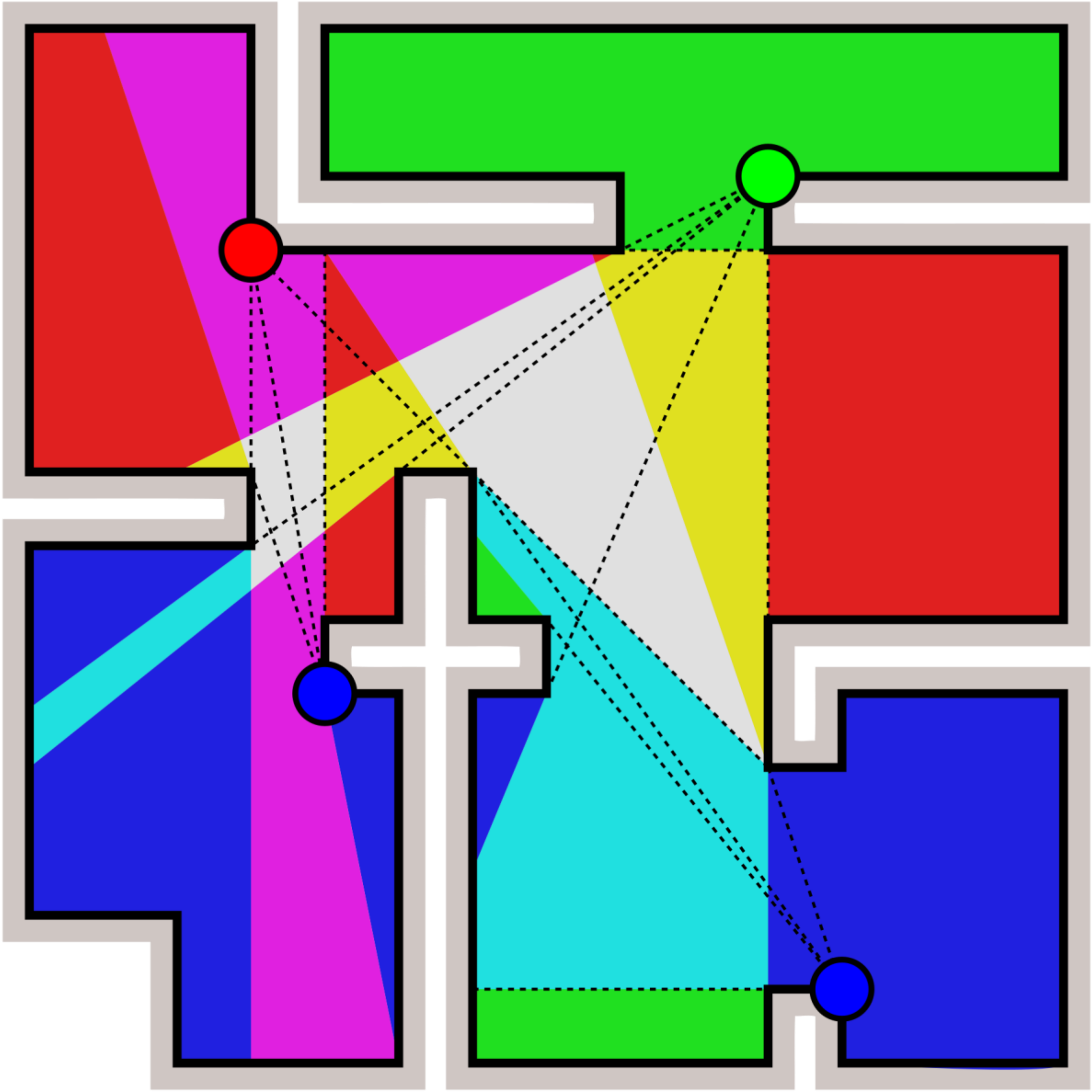




?!


흠....



2년 뒤, 1975년에 논문을 발표하여 증명

증명 완료!

2년 뒤, 1975년에 논문을 발표하여 증명

정답은....

2년 뒤, 1975년에 논문을 발표하여 증명

$\left\lfloor \frac{n}{3} \right\rfloor$명이면 충분하다.

2년뒤, 1975년에 논문을 발표하여 증명

꼭짓점의 개수 n에 대한 수학적 귀납법을 설계하고,
n = 3, 4, 5 인 경우에는 볼록한 모양 밖에 없으므로 끝,
n 보다 꼭짓점이 작은 모든 다각형에 대해 $\left\lfloor \frac{k}{3} \right\rfloor$ 명으로
감시가 다 가능하다고 할 때, 다각형 P를 P_1, P_2로
적절히 잘라 가정을 적용하여 $\left\lfloor \frac{k}{3} \right\rfloor$ 명으로 감시가
가능함을 증명하였다....

하지만 다각형을 적절히 자르는 방식에서 수학적 귀납법이 깨질 수 있어, 부등식을 유지하면서도 적절히 자르는 방법이 존재한다는 것을 따로 증명

꼭짓점의 개수 n에 대한 수학적 귀납법을 설계하고,
n = 3, 4, 5 인 경우에는 볼록한 모양 밖에 없으므로 끝,
n 보다 꼭짓점이 작은 모든 다각형에 대해 $\left\lfloor \frac{k}{3} \right\rfloor$ 명으로
감시가 다 가능하다고 할 때, 다각형 P를 P_1, P_2로
적절히 잘라 가정을 적용하여 $\left\lfloor \frac{k}{3} \right\rfloor$ 명으로 감시가
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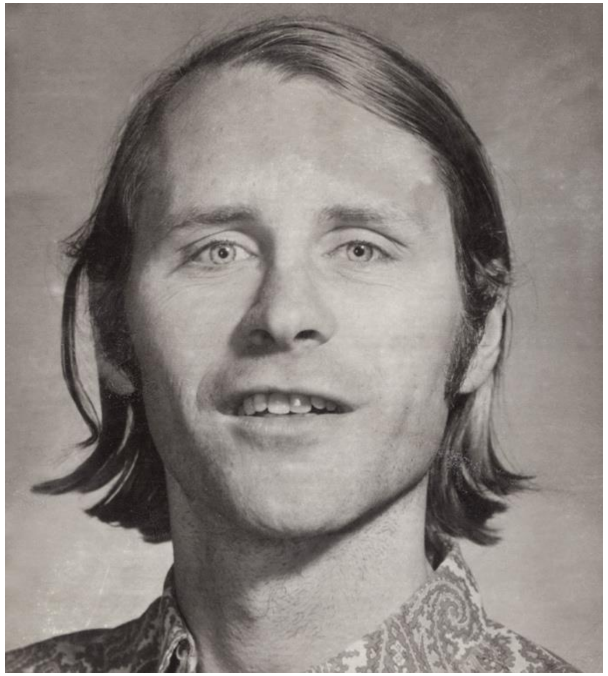
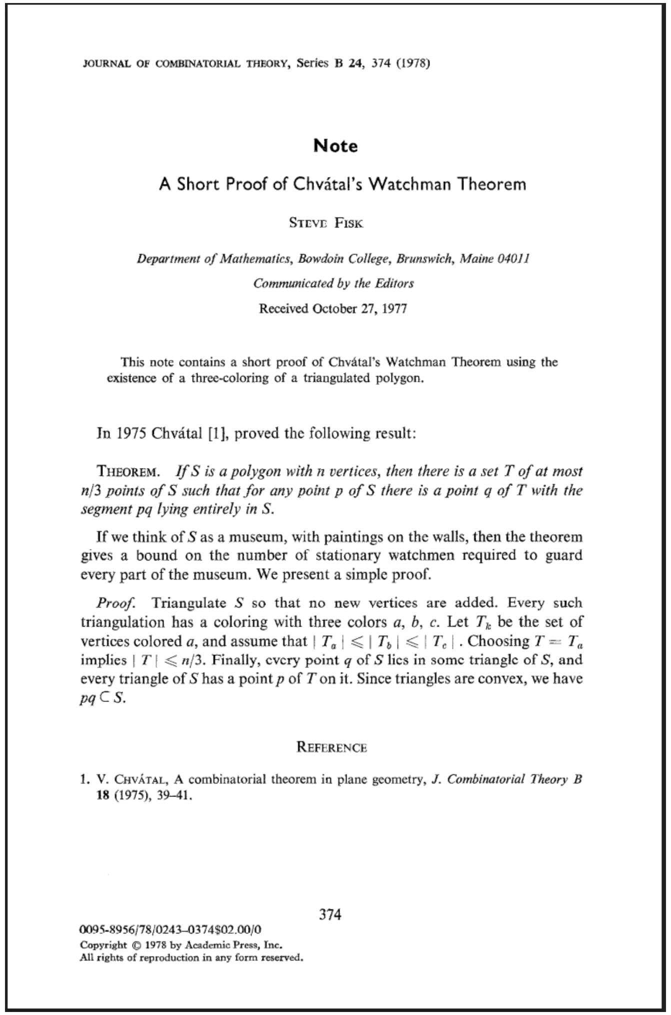
하지만 3년 뒤 1978년에 스티브 피스크(Steve Fisk)가 조합론적 방법을 통해 반 페이지로 이 문제를 증명

쉽던데 ㅋㅋ



스티브 피스크(Steve Fisk)



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Note

A Short Proof of Chvátal's Watchman Theorem

STEVE FISK

Department of Mathematics, Bowdoin College, Brunswich, Maine 04011

Communicated by the Editors

Received October 27, 1977

This note contains a short proof of Chvátal's Watchman Theorem using the existence of a three-coloring of a triangulated polygon.

In 1975 Chvátal [1], proved the following result:

THEOREM. *If S is a polygon with n vertices, then there is a set T of at most $n/3$ points of S such that for any point p of S there is a point q of T with the segment pq lying entirely in S.*

If we think of S as a museum, with paintings on the walls, then the theorem gives a bound on the number of stationary watchmen required to guard every part of the museum. We present a simple proof.

Proof. Triangulate S so that no new vertices are added. Every such triangulation has a coloring with three colors a, b, c. Let T_k be the set of vertices colored a, and assume that $|T_a| \leqslant |T_b| \leqslant |T_c|$. Choosing $T = T_a$ implies $|T| \leqslant n/3$. Finally, every point q of S lies in some triangle of S, and every triangle of S has a point p of T on it. Since triangles are convex, we have $pq \subset S$.

REFERENCE

1. V. CHVÁTAL, A combinatorial theorem in plane geometry, *J. Combinatorial Theory B* **18** (1975), 39–41.

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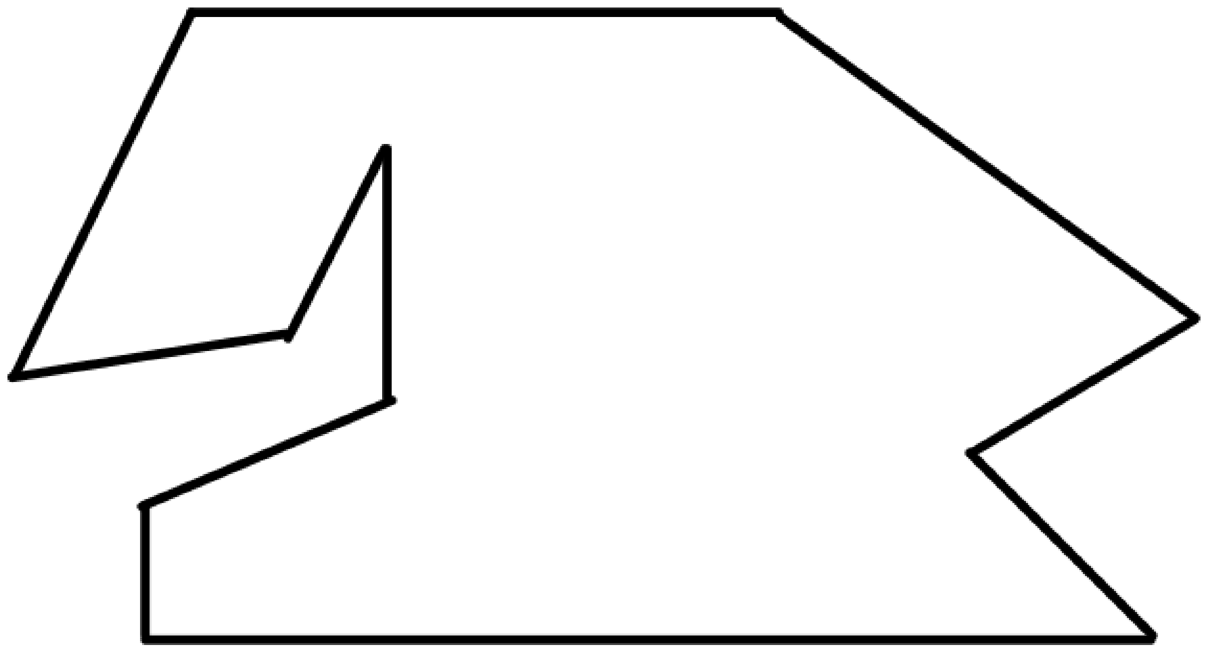
Fisk의 증명

1. 다각형 내부를 모두 삼각형으로 쪼갬



꼭짓점이 11개인 다각형

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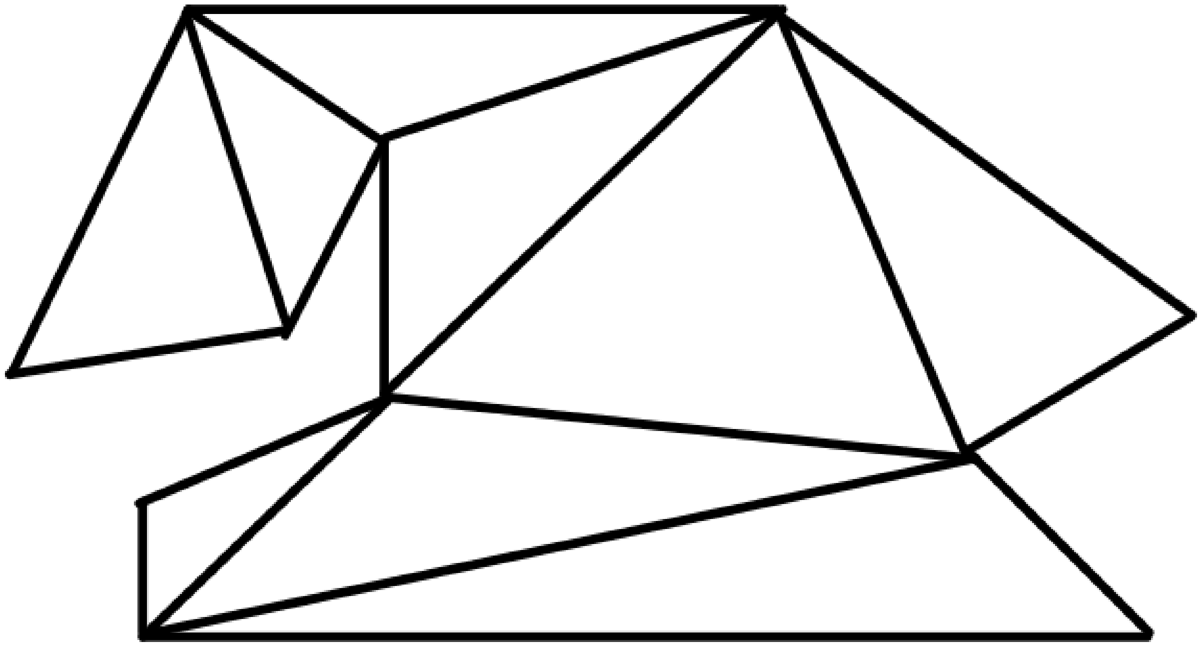
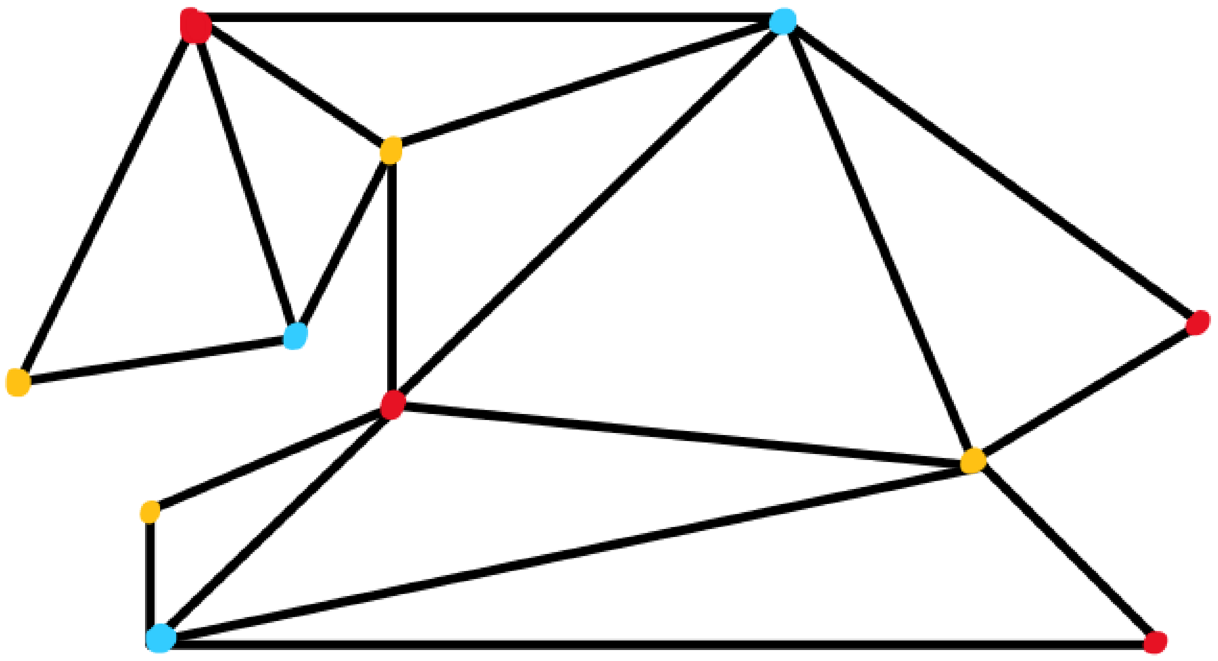


어떻게 쪼개든 상관 없음

Fisk의 증명

2. 삼각형의 꼭짓점을 3색으로 모두 다 다르게 칠함



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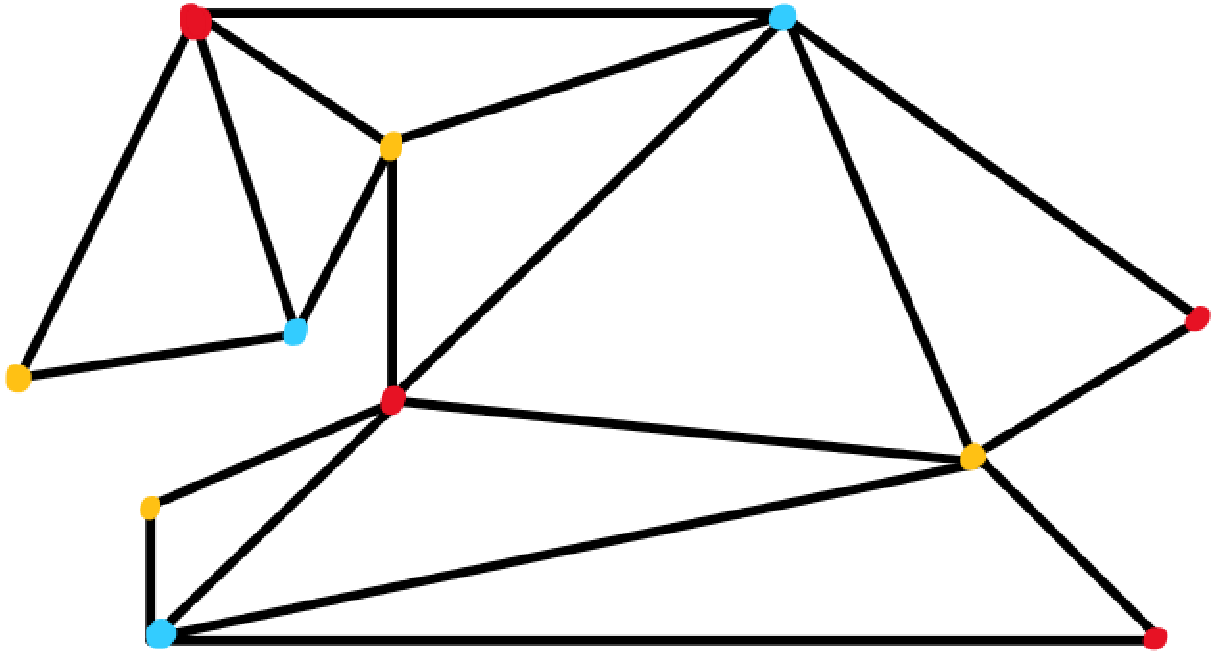

비둘기 집의 원리에 따라, $\left\lfloor \frac{11}{3} \right\rfloor = 3$ 개로 칠한 색이 적어도 하나 존재

Fisk의 증명

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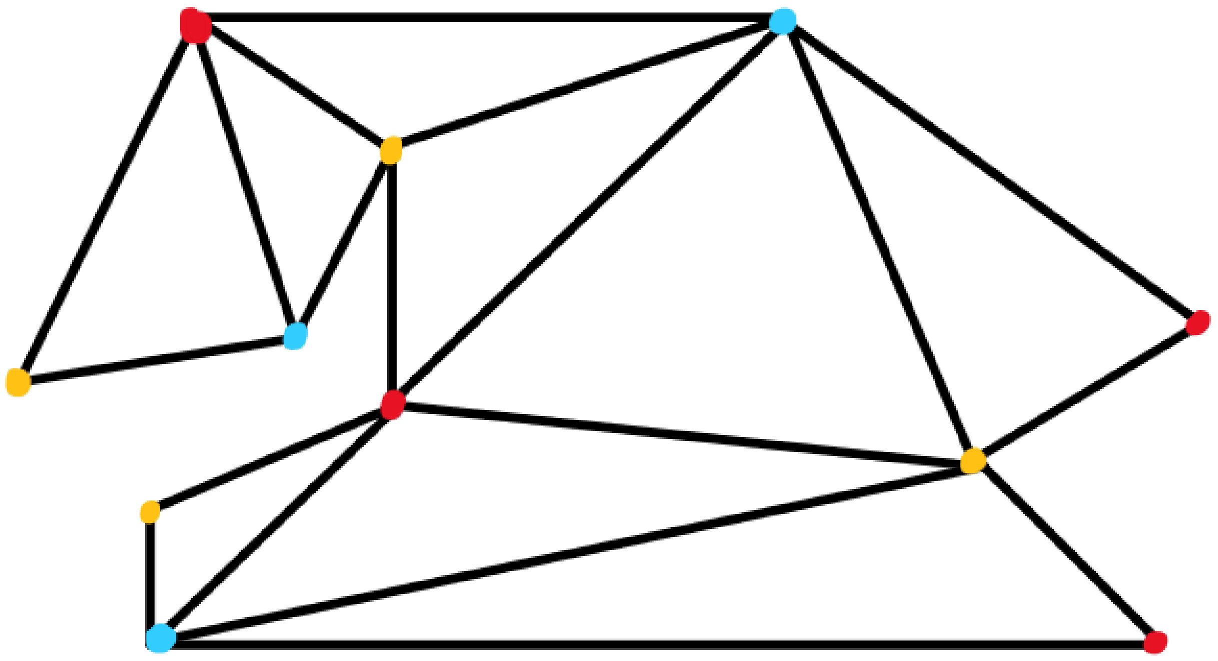
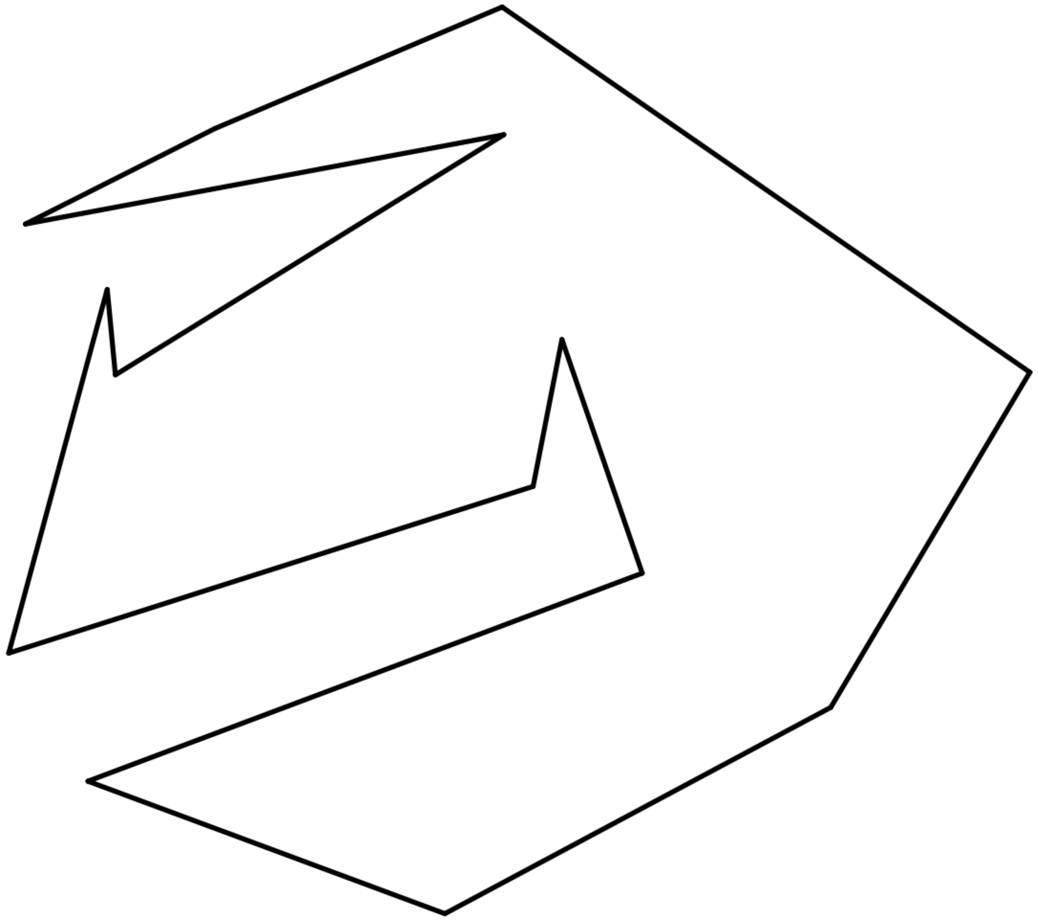


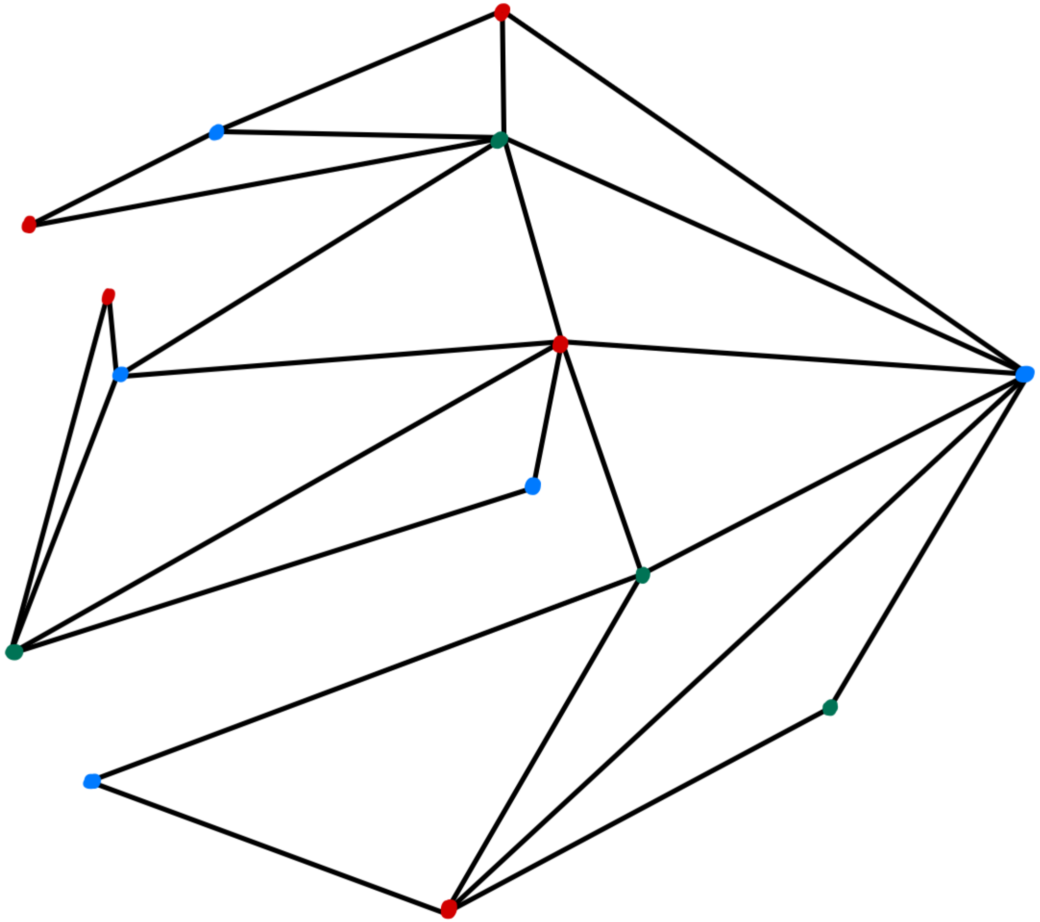
파란 색 점은 모든 삼각형의 꼭짓점 하나에 대응되고,
삼각형 내부는 볼록하므로 모두 다 감시 가능

또 다른 예시



꼭짓점이 14개인 다각형

또 다른 예시



$\left\lfloor \frac{14}{3} \right\rfloor = 4$ 명으로 모두 감시 가능

이후 연구

- 모든 내각이 90도 또는 270도인 경우, $\left\lfloor \frac{n}{4} \right\rfloor$ 명으로 모두 감시 가능

- 미술관이 내부에 구멍이 h개 있는 경우, $\left\lfloor \frac{n+2h}{3} \right\rfloor$ 명으로 모두 감시 가능

- 특정 도면이 주어졌을 때, 필요한 경비원 수와 좌표를 출력하는 알고리즘은 NP-hard 임이 증명됨

- 3차원 다면체의 경우, 자이델(Seidel)이라는 수학자가 모든 꼭짓점에 경비원을 세워도 다면체 내부 공간을 모두 감시하는 것이 불가능한 다면체가 있음을 증명함

4. 내쉬 균형
(Equilibrium Points in n-Person Games)

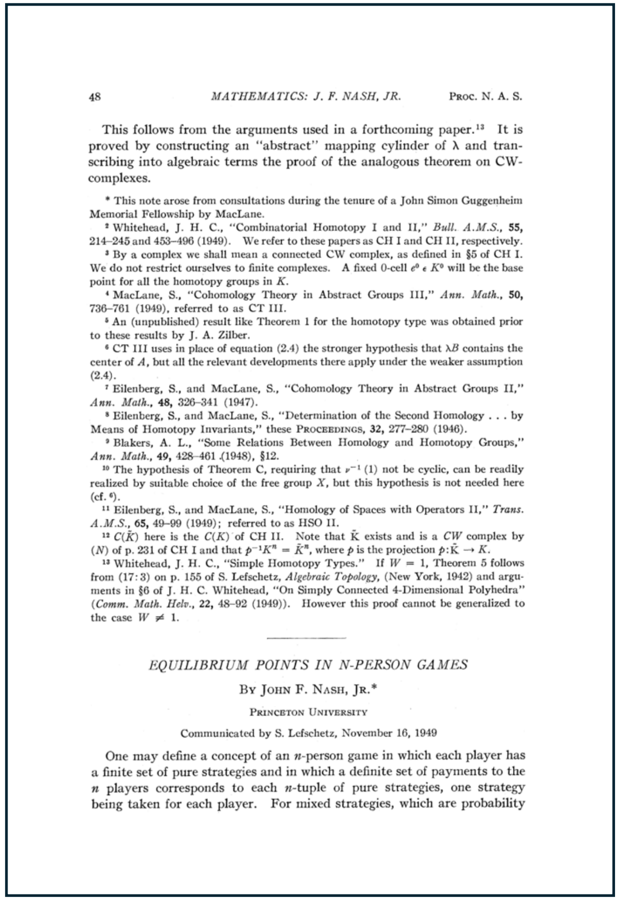
4. 내쉬 균형 (Equilibrium Points in n-Person Games)

(많이 어려움)

4. 내쉬 균형
(Equilibrium Points in n-Person Games)

(엄청난 비약과 수학적 오류가 있을 수 있음)

내쉬 균형



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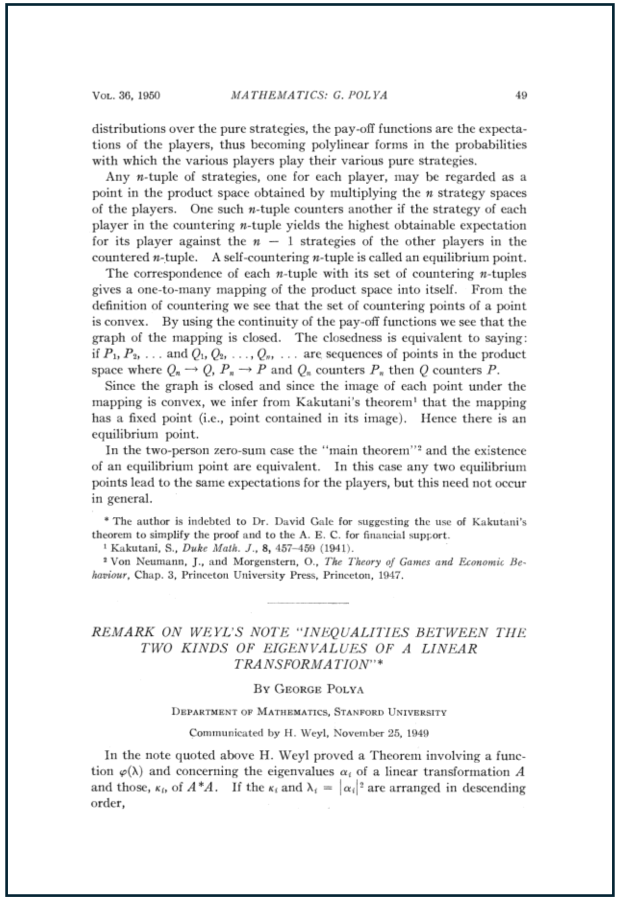
EQUILIBRIUM POINTS IN N-PERSON GAMES

BY JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability



distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \to Q$, $P_n \to P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

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*REMARK ON WEYL'S NOTE "INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION"**

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단 2페이지로 논문 작성(내용은 더 짧음)

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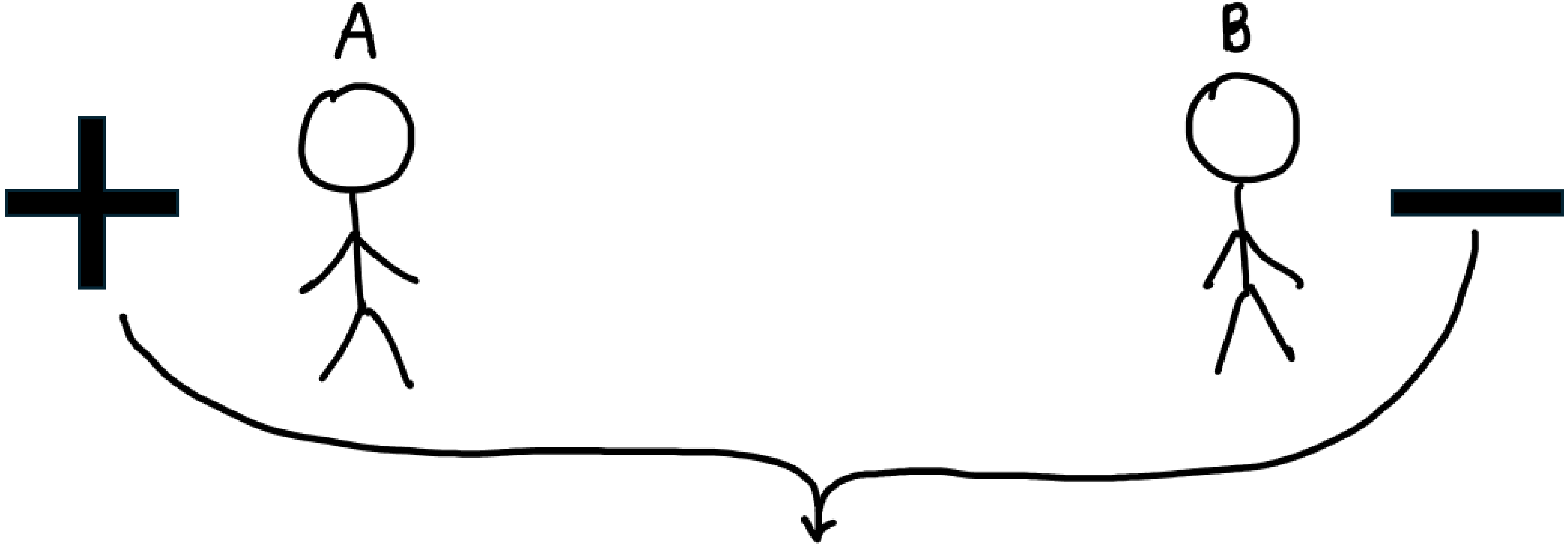
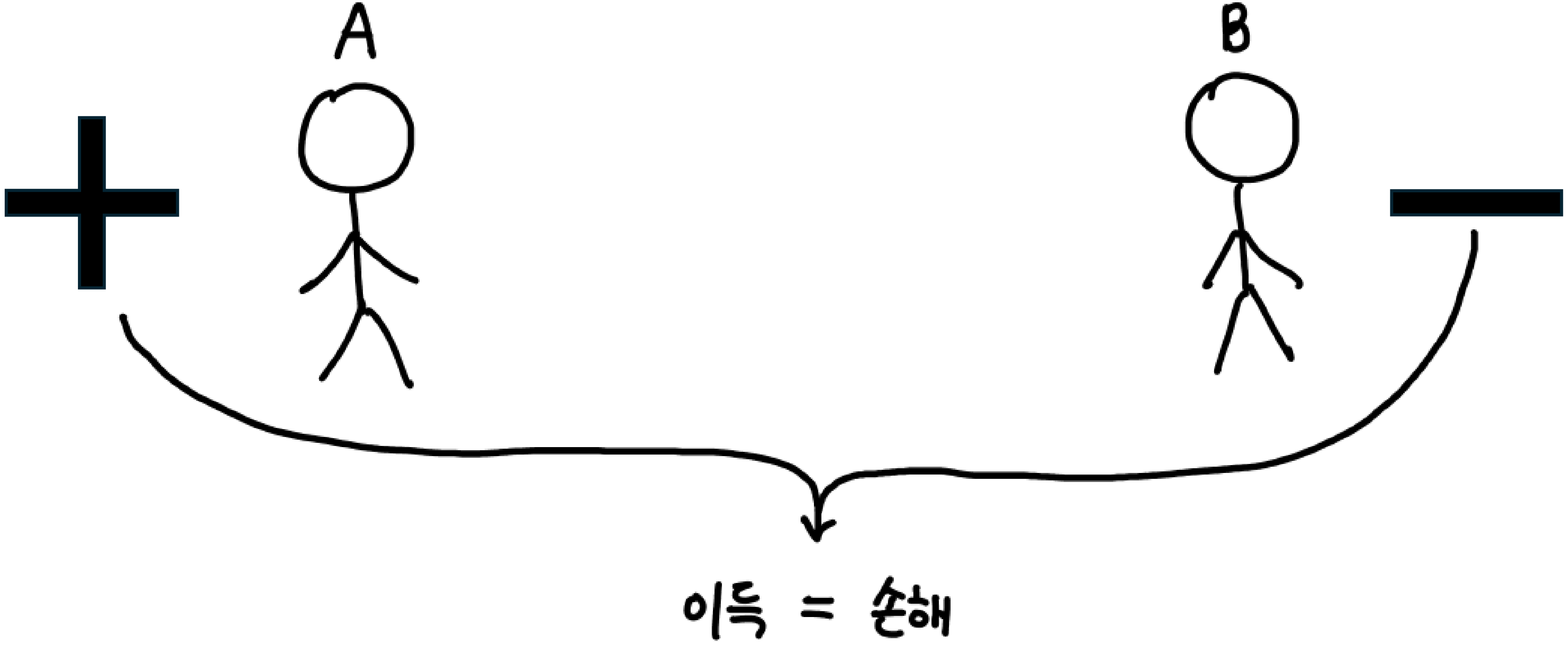


이득 = 손해

제로섬

Ex) 도박, 가위바위보, 선물 거래



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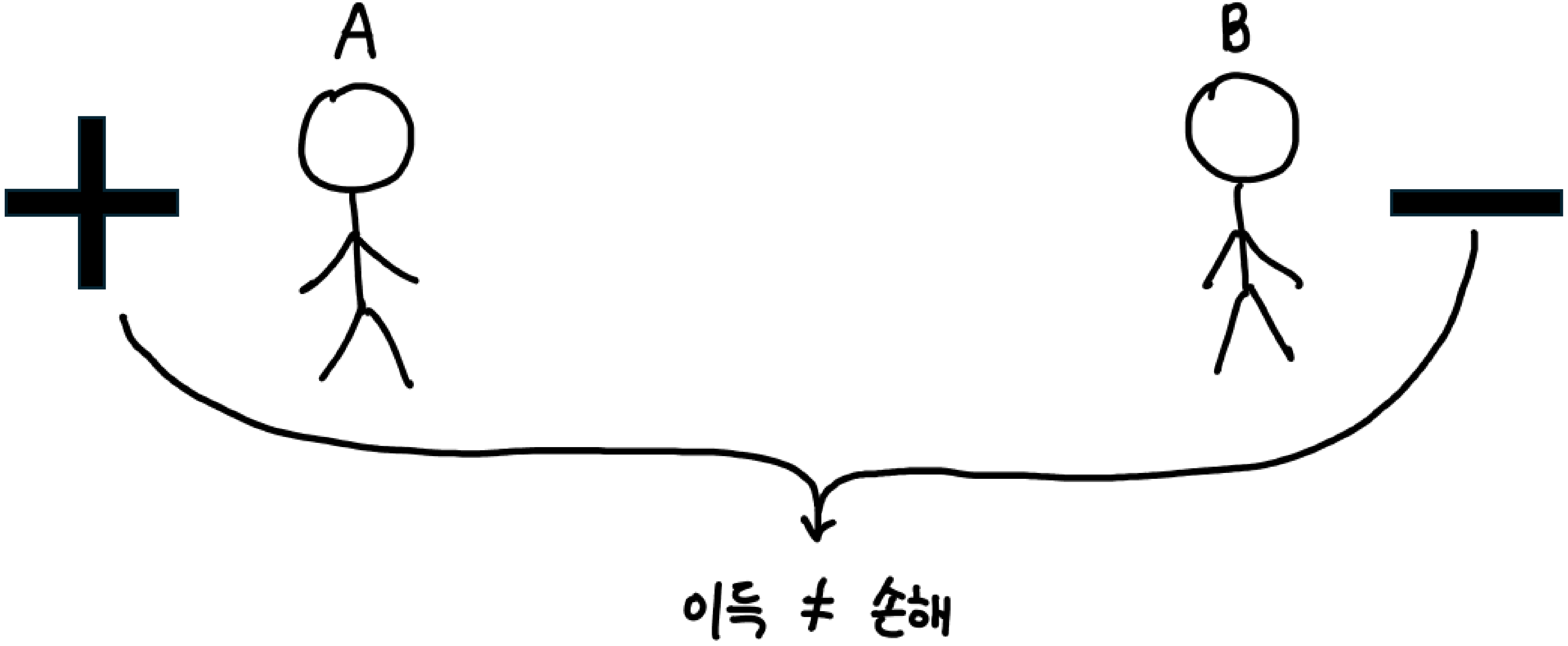
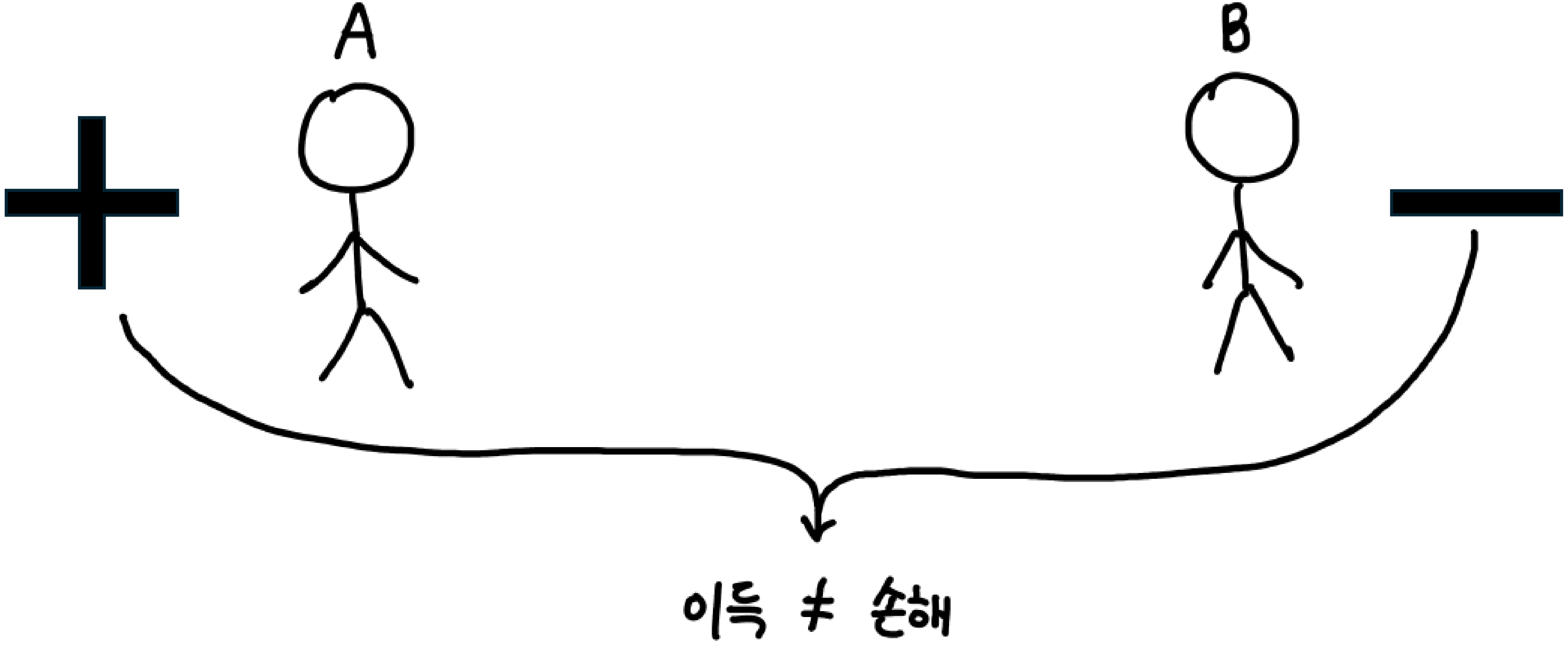
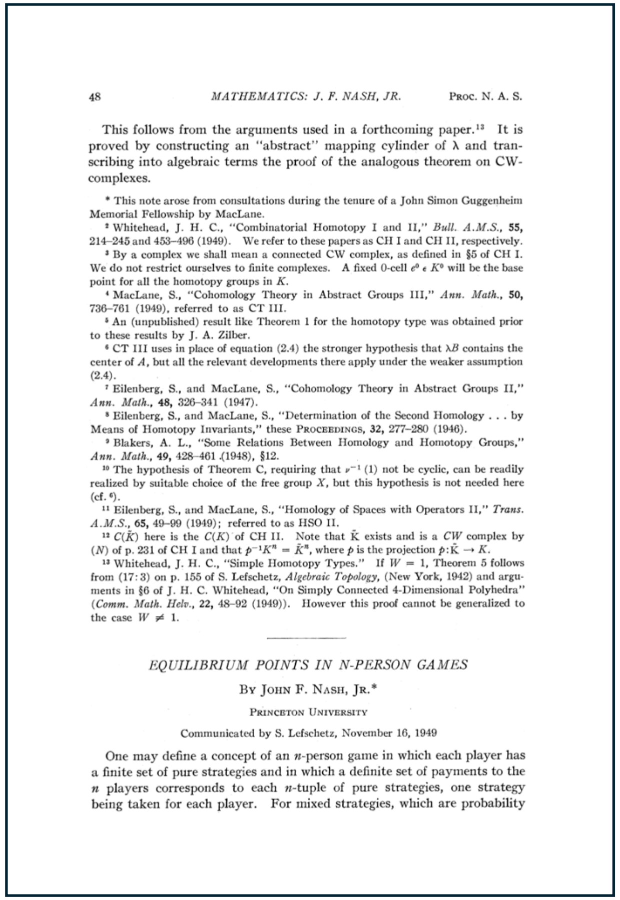


논제로섬

Ex) 무역, 기술 혁신, 주식



내쉬 균형



48 *MATHEMATICS: J. F. NASH, JR.* PROC. N. A. S.

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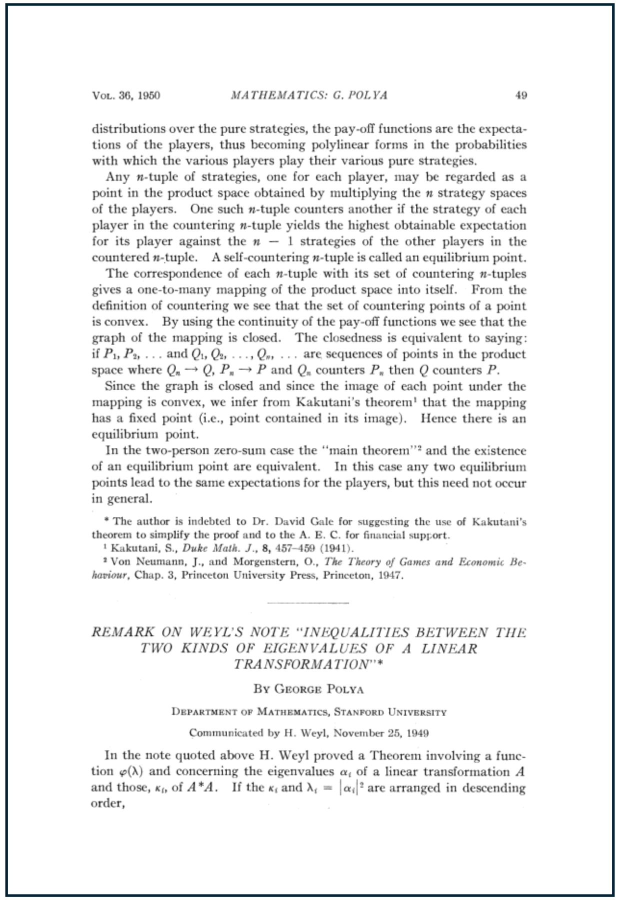

VOL. 36, 1950 *MATHEMATICS: G. POLYA* 49

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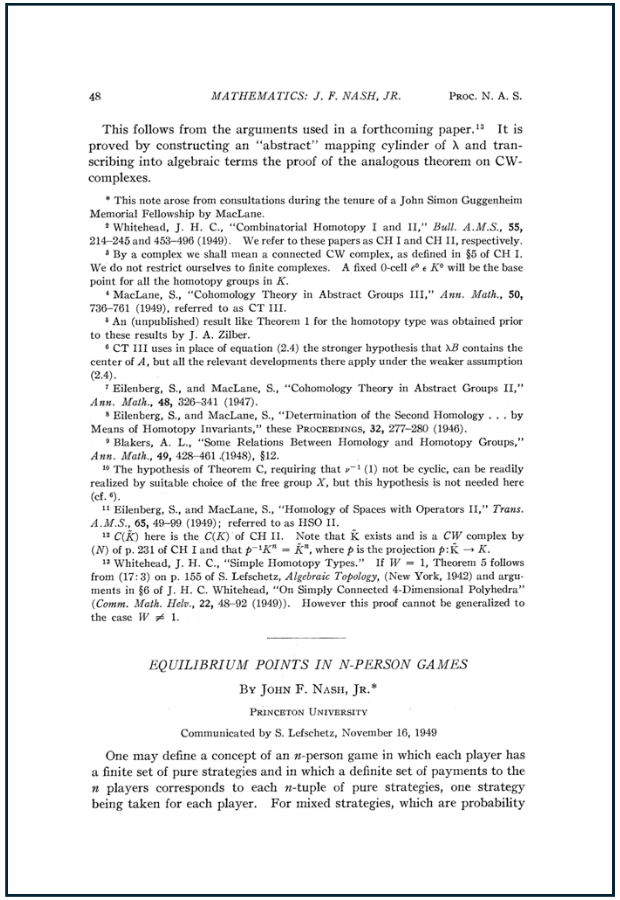
내쉬 균형



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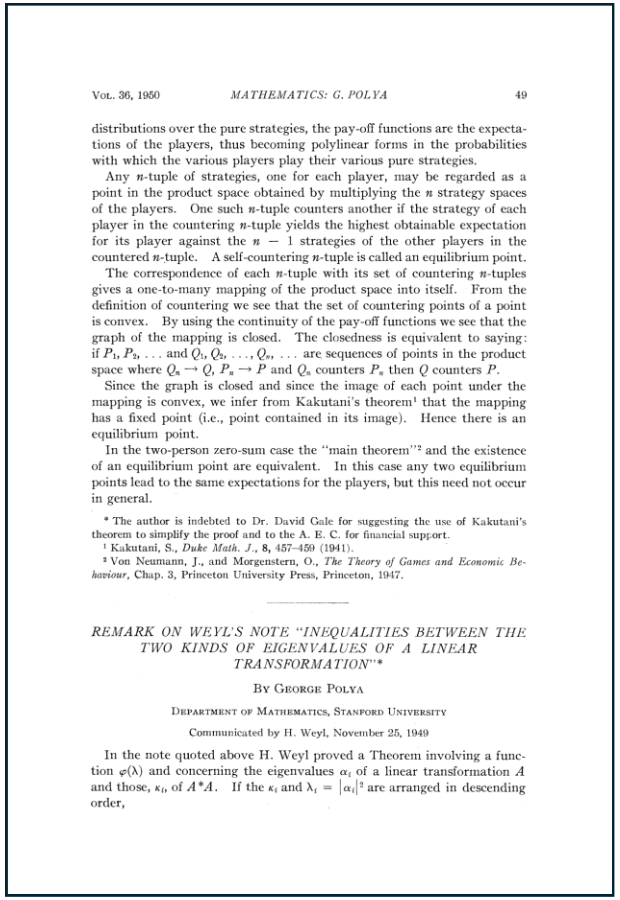
One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability



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distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \to Q$, $P_n \to P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

* The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

[1] Kakutani, S., *Duke Math. J.*, **8,** 457–459 (1941).

[2] Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

*REMARK ON WEYL'S NOTE "INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION"**

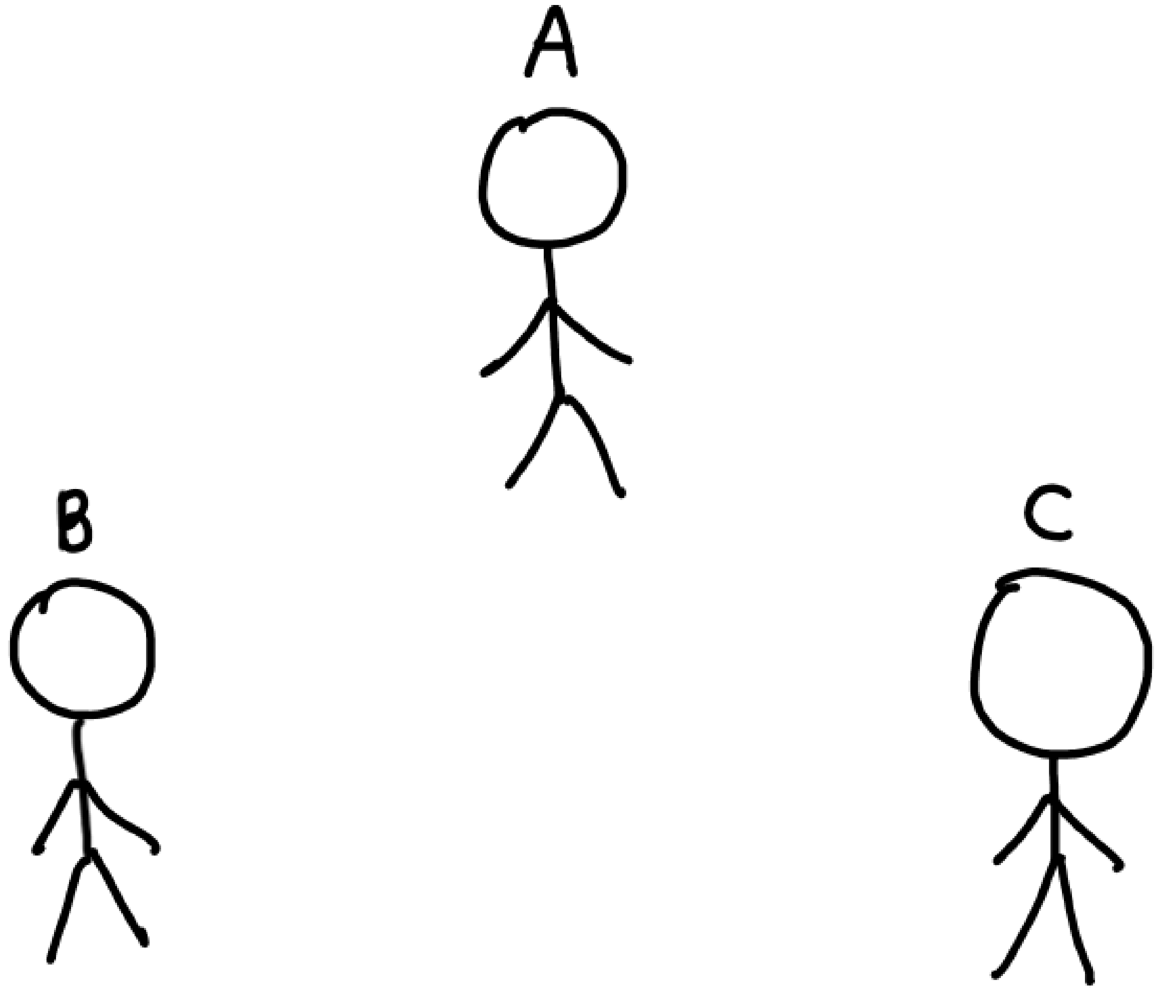
BY GEORGE POLYA

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY

Communicated by H. Weyl, November 25, 1949

In the note quoted above H. Weyl proved a Theorem involving a function $\varphi(\lambda)$ and concerning the eigenvalues α_i of a linear transformation A and those, κ_i, of A^*A. If the κ_i and $\lambda_i = |\alpha_i|^2$ are arranged in descending order,

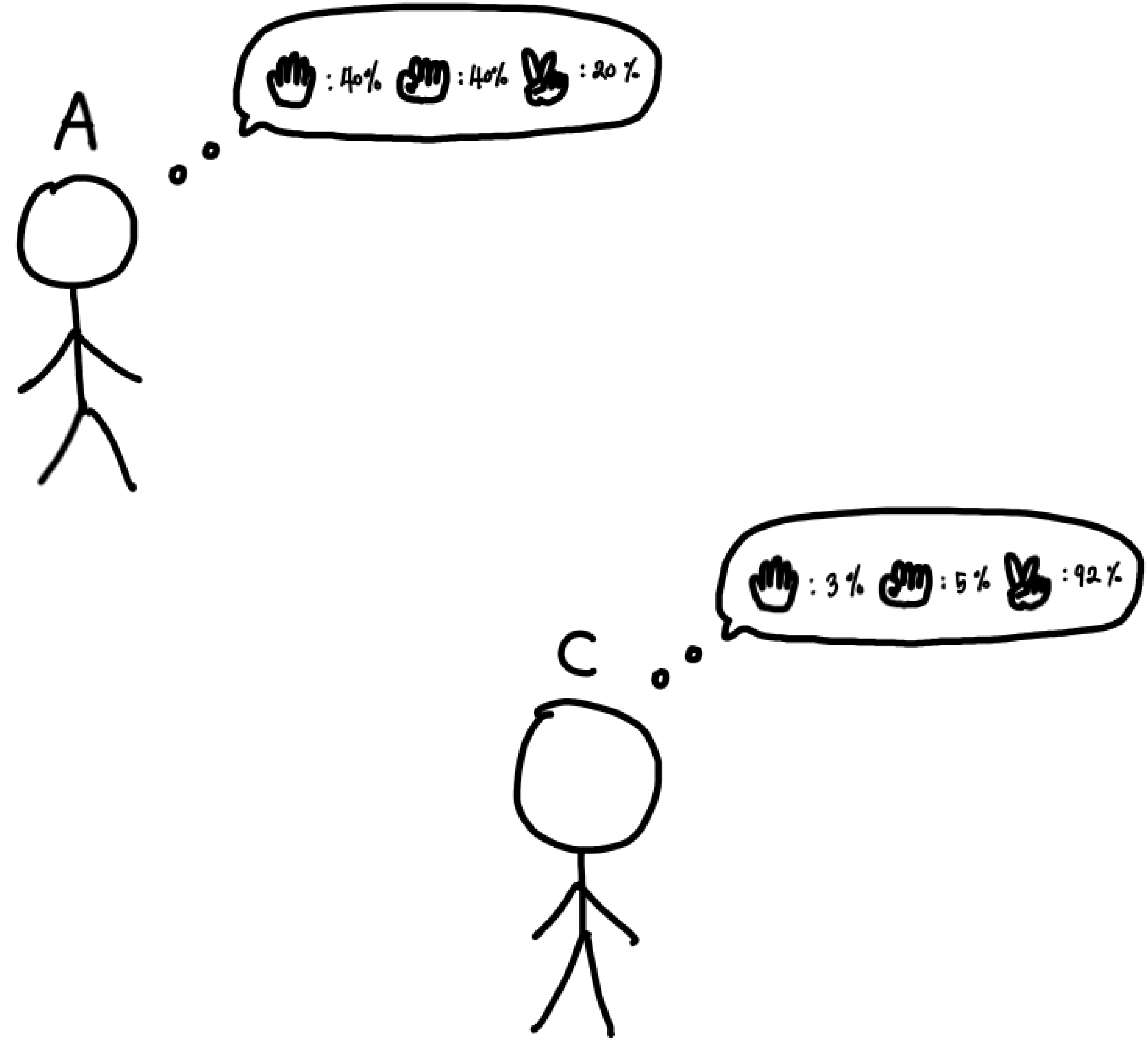
'논제로섬 게임'에서도 균형점이 존재한다는 것을 존 내쉬(John Nash)가 증명

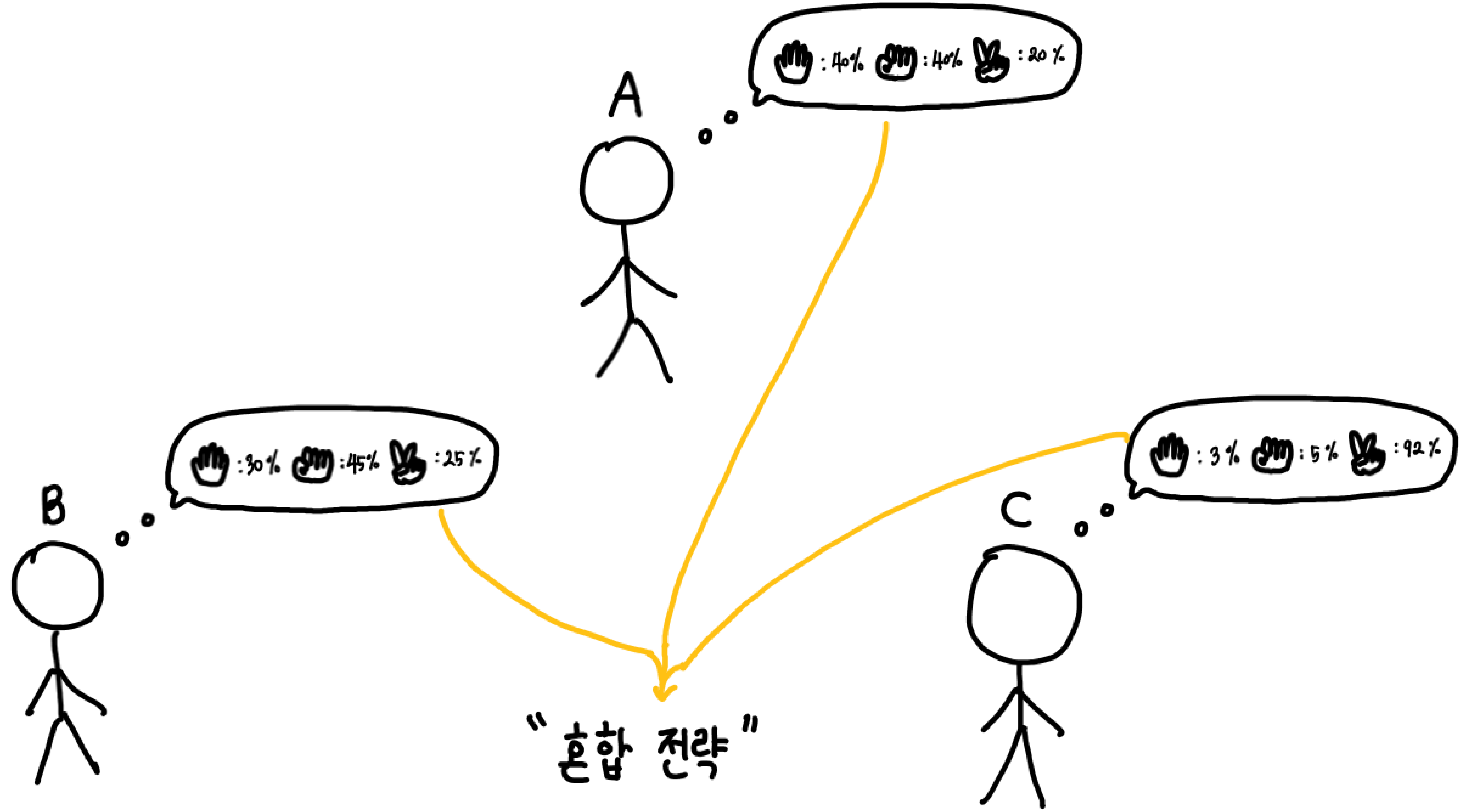


세 명의 사람이

가위바위보를 한다고

가정


A
: 40% : 40% : 20%
B
: 30% : 45% : 25%
C
: 3% : 5% : 92%



A
: 40% : 40% : 20%
B
: 30% : 45% : 25%
C
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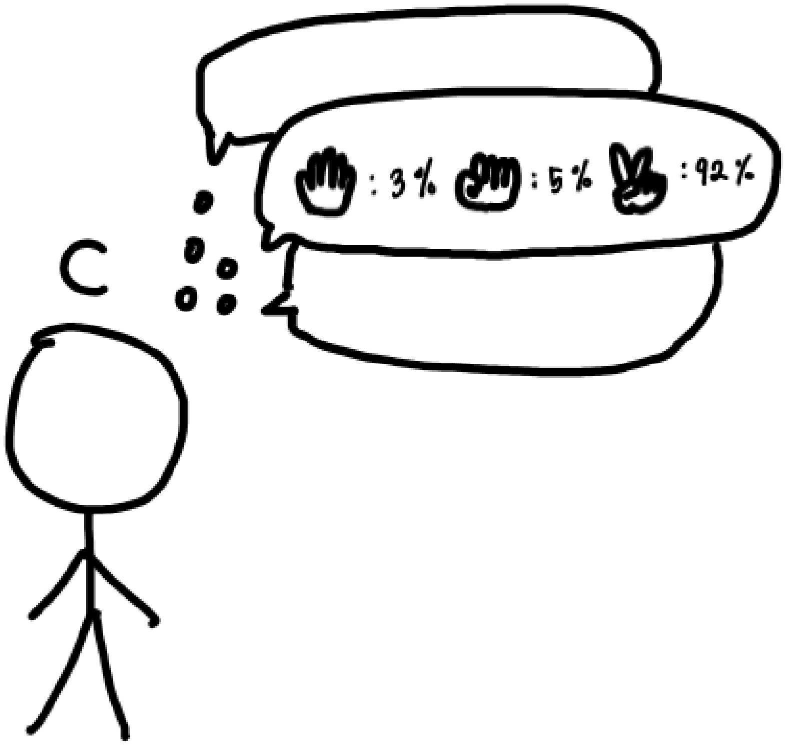
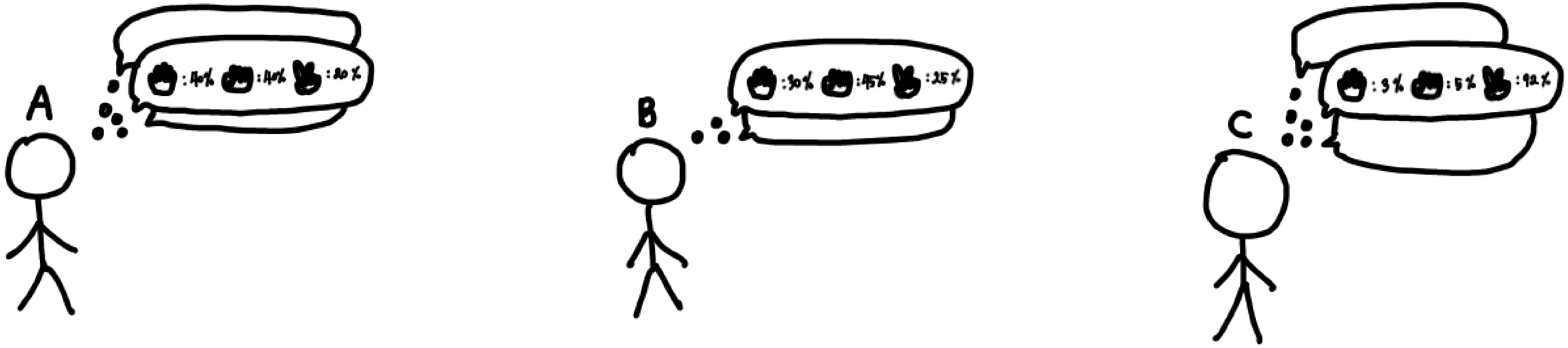
"혼합 전략"

각 사람마다 혼합전략을 유한 개 갖고 있을 수 있음.

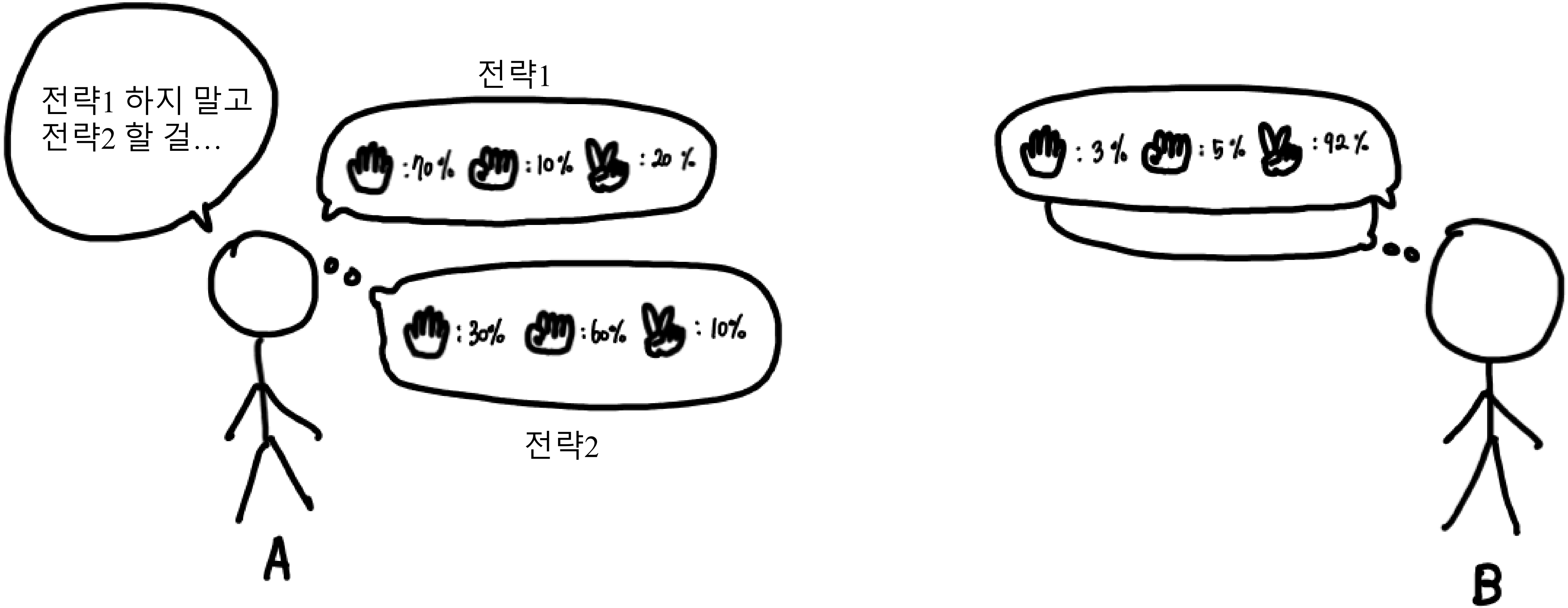


그런데 내쉬 균형이란 무엇일까?



모두가 자신의 혼합전략을 공개하였을 때, 지금 고른 전략이 본인에게 최선이라고 느끼는 상태

그런데 내쉬 균형이란 무엇일까?



⇒ 내쉬 균형 X

최선이라고 느끼는 상태 = 후회를 하지 않는 상태

수학적 모델링

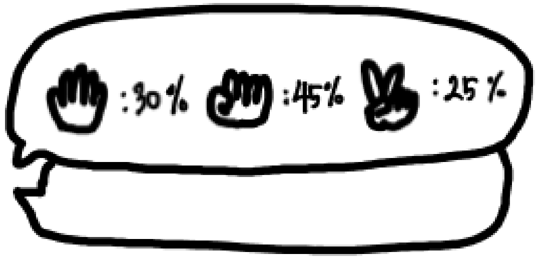
A :

$\Rightarrow \sigma_{A1} = (0.4, 0.4, 0.2) \in \mathbb{R}^3$

$\Rightarrow \sigma_{A2}$

$\Rightarrow \sigma_{A3}$

B :

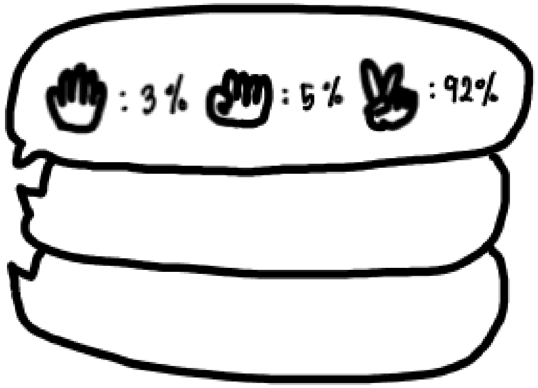


$\Rightarrow \sigma_{B1} = (0.3, 0.45, 0.25) \in \mathbb{R}^3$

$\Rightarrow \sigma_{B2}$

"σ_{A1}": A의 첫 번째 전략

C :



$\Rightarrow \sigma_{C1} = (0.03, 0.05, 0.92) \in \mathbb{R}^3$

$\Rightarrow \sigma_{C2}$

$\Rightarrow \sigma_{C3}$

Δ_A = { A의 모든 혼합전략 } = { $\sigma_{A1}, \sigma_{A2}, \sigma_{A3}$ }

Δ_B = { B 〃 } = { σ_{B1}, σ_{B2} }

Δ_C = { C 〃 } = { $\sigma_{C1}, \sigma_{C2}, \sigma_{C3}$ }

"Δ_i : i의 전략공간"

A, B, C 가 동시에 취할 수 있는 전략의 경우의 수

$$\Rightarrow \left\{ \begin{array}{l} (\sigma_{A1}, \sigma_{B1}, \sigma_{C1}), (\sigma_{A2}, \sigma_{B1}, \sigma_{C1}), \\ (\sigma_{A2}, \sigma_{B2}, \sigma_{C1}), (\sigma_{A2}, \sigma_{B2}, \sigma_{C2}), \cdots \end{array} \right\} = \Sigma .$$

Σ : 전체 게임의 상태 공간

$$T : \Sigma \to \Sigma, \qquad c_{i\alpha} = \max(0, p_{i\alpha}(s) - p_i(s))$$

함수를 정의, 전략을 바꿨을 때 오르는 점수를 '추가 이득'이라고 부를 때, 추가 이득이 클수록 그 전략을 선택할 확률을 높여줌.

$c_{i\alpha}$: 전략을 바꿨을 때의 추가이득

$p_{i\alpha}$: 전략을 α로 바꿨을 때의 수익, p_i : 현재 수익

$$T : \Sigma \rightarrow \Sigma, \qquad c_{i\alpha} = \max(0, p_{i\alpha}(s) - p_i(s))$$

Σ는 위상수학에서 'convex' 와 'compact' 라고 불리는 매우 좋은 성질을 가진 공간, T는 Σ 상에서 연속 함수

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=> ‘Brouwer Fixed-Point Theorem’에 의해 $T(s) = s$를 만족시키는 s가 존재.

$$T : \Sigma \to \Sigma, \qquad c_{i\alpha} = \max(0, p_{i\alpha}(s) - p_i(s))$$

전략을 바꿨을 때 추가 이득이 클수록
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s에서는 함수를 거쳐도 확률이 변하지 않음

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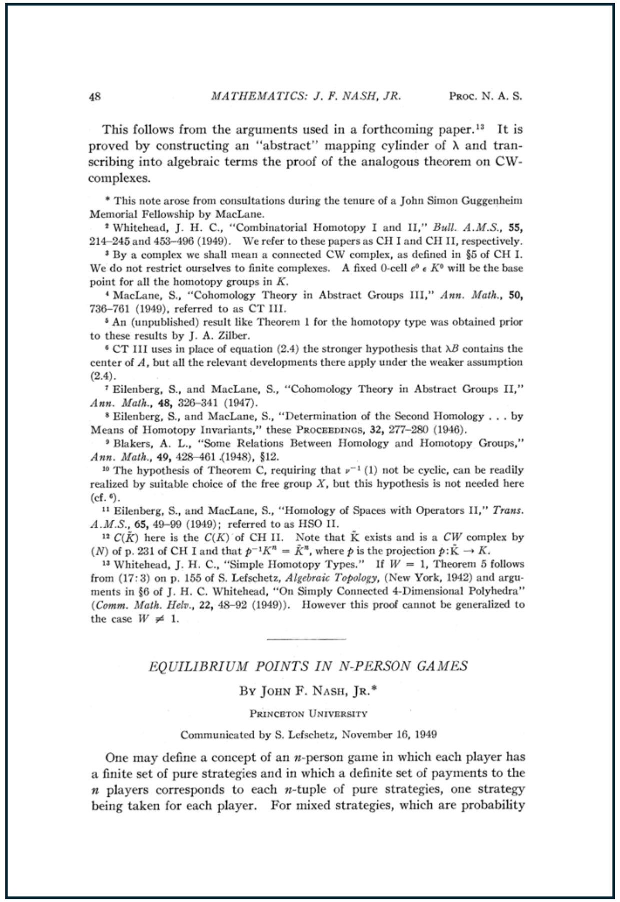
⇒ 어떤 전략으로 바꾸더라도 현재보다 더 나은
수익을 얻을 수 없음을 의미함, 즉 내쉬 균형


48 MATHEMATICS: J. F. NASH, JR. PROC. N. A. S.

This follows from the arguments used in a forthcoming paper.[13] It is proved by constructing an "abstract" mapping cylinder of λ and transcribing into algebraic terms the proof of the analogous theorem on CW-complexes.

* This note arose from consultations during the tenure of a John Simon Guggenheim Memorial Fellowship by MacLane.

[2] Whitehead, J. H. C., "Combinatorial Homotopy I and II," *Bull. A.M.S.*, **55,** 214–245 and 453–496 (1949). We refer to these papers as CH I and CH II, respectively.

[3] By a complex we shall mean a connected CW complex, as defined in §5 of CH I. We do not restrict ourselves to finite complexes. A fixed 0-cell $e^0 \epsilon K^0$ will be the base point for all the homotopy groups in K.

[4] MacLane, S., "Cohomology Theory in Abstract Groups III," *Ann. Math.*, **50,** 736–761 (1949), referred to as CT III.

[5] An (unpublished) result like Theorem 1 for the homotopy type was obtained prior to these results by J. A. Zilber.

[6] CT III uses in place of equation (2.4) the stronger hypothesis that λB contains the center of A, but all the relevant developments there apply under the weaker assumption (2.4).

[7] Eilenberg, S., and MacLane, S., "Cohomology Theory in Abstract Groups II," *Ann. Math.*, **48,** 326–341 (1947).

[8] Eilenberg, S., and MacLane, S., "Determination of the Second Homology . . . by Means of Homotopy Invariants," these PROCEEDINGS, **32,** 277–280 (1946).

[9] Blakers, A. L., "Some Relations Between Homology and Homotopy Groups," *Ann. Math.*, **49,** 428–461 (1948), §12.

[10] The hypothesis of Theorem C, requiring that $\nu^{-1}(1)$ not be cyclic, can be readily realized by suitable choice of the free group X, but this hypothesis is not needed here (cf. [6]).

[11] Eilenberg, S., and MacLane, S., "Homology of Spaces with Operators II," *Trans. A.M.S.*, **65,** 49–99 (1949); referred to as HSO II.

[12] $C(\tilde{K})$ here is the $C(K)$ of CH II. Note that $\tilde{K}$ exists and is a CW complex by (N) of p. 231 of CH I and that $p^{-1}K^n = \tilde{K}^n$, where p is the projection $p: \tilde{K} \to K$.

[13] Whitehead, J. H. C., "Simple Homotopy Types." If $W = 1$, Theorem 5 follows from (17:3) on p. 155 of S. Lefschetz, *Algebraic Topology*, (New York, 1942) and arguments in §6 of J. H. C. Whitehead, "On Simply Connected 4-Dimensional Polyhedra" (*Comm. Math. Helv.*, **22,** 48–92 (1949)). However this proof cannot be generalized to the case $W \neq 1$.

EQUILIBRIUM POINTS IN N-PERSON GAMES

BY JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Lefschetz, November 16, 1949

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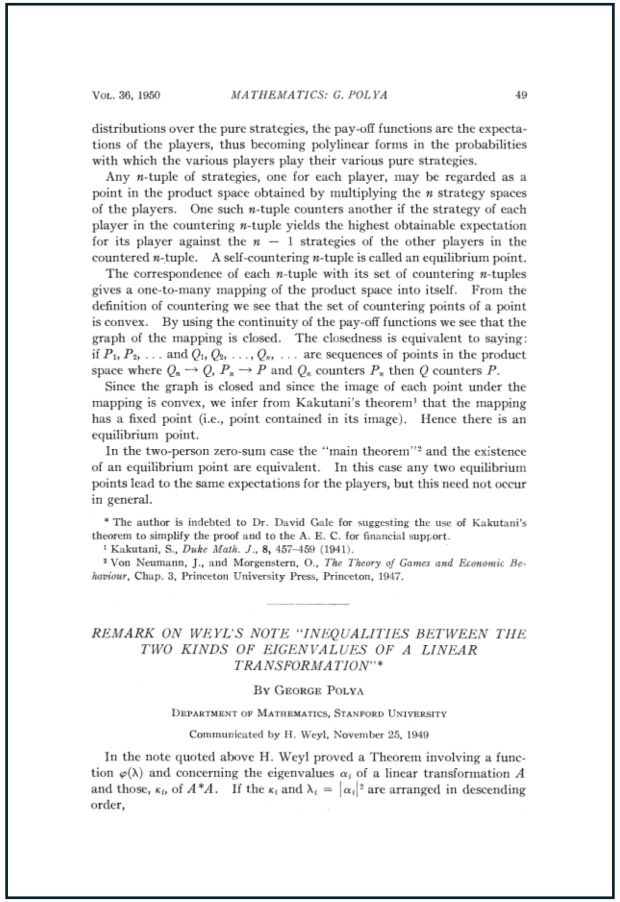
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distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

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경제학의 근본적인 질문을 '위상 수학'을 사용하여 해결

내쉬 균형을 더 엄밀하게 알고 싶다면?

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내쉬 균형 검색

끝

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