

게임은 수학이다

Uniqueness of the Sudoku Solution

2024310862 이재웅

차례 Contents

- 게임 소개
- 풀이
 - 정의
 - 보조정리
 - 기타 과정
- 컴퓨터 계산
- 결론

게임 소개 Introduction

- Sudoku
- Originated from the Latin Square by Leonhard Euler
 - each symbol occurs exactly once in each row and each column
+ in each box

1	3	2	4
4	2	3	1
3	4	1	2
2	1	4	3

게임 소개 Introduction

- Sudoku
- 4×4, 6×6, 9×9 etc.

1	3	2	4
4	2	3	1
3	4	1	2
2	1	4	3

2	5	6	4	1	3
4	3	1	2	6	5
1	2	5	6	3	4
6	4	3	1	5	2
5	1	2	3	4	6
3	6	4	5	2	1

6	1	9	7	4	2	3	5	8
4	5	7	8	3	1	6	9	2
8	2	3	6	9	5	1	7	4
2	3	6	4	1	9	5	8	7
5	9	1	2	7	8	4	6	3
7	8	4	3	5	6	2	1	9
9	6	2	1	8	3	7	4	5
3	4	5	9	6	7	8	2	1
1	7	8	5	2	4	9	3	6

질문 Question

- Does every Sudoku have a unique solution?
- Is this related to the number of empty cells?

							1	
					2			3
			4					
						5		
4		1	6					
		7	1					
	5					2		
				8			4	
	3		9	1				

with 64 blanks
only 1 solution

6			7	4	2	3	5	8
4	5	7	8	3	1	6	9	2
8	2	3	6	9	5	1	7	4
2	3	6	4	1	9	5	8	7
5			2	7	8	4	6	3
7	8	4	3	5	6	2	1	9
9	6	2	1	8	3	7	4	5
3	4	5	9	6	7	8	2	1
1	7	8	5	2	4	9	3	6

with 4 blanks
multiple solutions

질문 Question

- Does every Sudoku have a unique solution?
- Is this related to the number of empty cells?
- What is the minimum clues for a unique solution?

풀이 Solution

- The number of completed 9×9 Sudoku grids is

$$6,670,903,752,021,072,936,960 \approx 6.7 \times 10^{21}$$

해 경 조 억 만

- The number of equivalence class reduced by row/column symmetries and digit permutations is

$$5,472,730,538$$

풀이 Solution

- Possible cases of a 16-clue puzzles per grid are

$$\binom{81}{16} = 3.4 \times 10^{16}$$

- Find unavoidable sets
- Track 16-clue puzzles that hit all unavoidable sets

9	3	7	8	5	6	2	4	1
5	6	2	1	9	4	3	8	7
4	8	1	2	7	3	5	6	9
8	2	3	6	4	7	9	1	5
6	1	5	9	3	2	4	7	8
7	4	9	5	8	1	6	2	3
3	7	8	4	6	9	1	5	2
1	9	6	7	2	5	8	3	4
2	5	4	3	1	8	7	9	6

풀이 Solution - Definition

- A Sudoku solution grid could be represented a function

$$\{0, 1, 2, \dots, 80\} \rightarrow \{1, 2, \dots, 9\}$$

- Let G be a solution grid. (완성 스도쿠)
 - $G = \{\langle 0, 9 \rangle, \langle 1, 3 \rangle, \dots, \langle 80, 6 \rangle\}$
- A subset $X \subset G$ is called *an unavoidable set*, if $G \setminus X$ has multiple completions.
 - $X_1 := \{\langle 0, 9 \rangle, \langle 4, 5 \rangle, \langle 9, 5 \rangle, \langle 13, 9 \rangle\}$ (주황색 셀)
 - $G \setminus X_1$ has 2 solutions, X_1 is *an unavoidable set*.

9	3	7	8	5	6	2	4	1
5	6	2	1	9	4	3	8	7
4	8	1	2	7	3	5	6	9
8	2	3	6	4	7	9	1	5
6	1	5	9	3	2	4	7	8
7	4	9	5	8	1	6	2	3
3	7	8	4	6	9	1	5	2
1	9	6	7	2	5	8	3	4
2	5	4	3	1	8	7	9	6

풀이 Solution - Lemma

- Suppose that $X \subset G$ is a set of clues of a sudoku solution grid G , such that X hits (intersects) every unavoidable set of G .
- Then G is the only completion of X .
- G 는 완성 스도쿠, X 는 G 에서 몇 칸 비어있는 스도쿠
- X 가 G 의 모든 *unavoidable sets*와 공통원소를 가진다면, G 는 X 의 유일해이다.

풀이 Solution - Lemma

- Suppose that $X \subset G$ is a set of clues of a sudoku solution grid G , such that X hits (intersects) every unavoidable set of G .
- Then G is the only completion of X .

- $X_1 := \{\langle 0,9 \rangle, \langle 4,5 \rangle, \langle 9,5 \rangle, \langle 13,9 \rangle\}$ (주황색 셀)

$$X_2 := \{\langle 32,7 \rangle, \langle 34,1 \rangle, \langle 41,2 \rangle, \langle 43,7 \rangle, \\ \langle 50,1 \rangle, \langle 52,2 \rangle\} \text{ (초록색 셀)}$$

- $X := G \cap (X_1 \cup X_2)^c$

	3	7	8		6	2	4	1
	6	2	1		4	3	8	7
4	8	1	2	7	3	5	6	9
8	2	3	6	4		9		5
6	1	5	9	3		4		8
7	4	9	5	8		6		3
3	7	8	4	6	9	1	5	2
1	9	6	7	2	5	8	3	4
2	5	4	3	1	8	7	9	6

풀이 Solution - Lemma

- Suppose that $X \subset G$ is a set of clues of a sudoku solution grid G , such that X hits (intersects) every unavoidable set of G .
- Then G is the only completion of X .
- Pf) If X had multiple completions, then $G \setminus X$ would be an unavoidable set not hit by X , which is a contradiction.

풀이 Solution - Remove Duplicates

- There are still too many cases to check.

풀이 Solution

- The number of completed 9×9 Sudoku grids is

$$6,670,903,752,021,072,936,960 \approx 6.7 \times 10^{21}$$

해 경 조 억 만

- The number of equivalence class **reduced by row/column symmetries and digit permutations** is

$$5,472,730,538$$

풀이 Solution - Remove Duplicates

- There are still too many cases to check.

1		2	3	4				
4		3	1	5				
		5		2				

1		2	3		4			
4		3	1		5			
		5			2			

We count duplicates
regardless of their positions

풀이 Solution - Add Missing Cases

- Also, there could be *an unavoidable set* of degree 2

1			2			3		
2			3			1		
3			1			2		

Assume that the remaining cells are already filled.

풀이 Solution - Lemma

- Suppose that $X \subset G$ is a set of clues of a sudoku solution grid G , such that X hits (intersects) every unavoidable set of G .
- Then G is the only completion of X .
- G 는 완성 스도쿠, X 는 G 에서 몇 칸 비어있는 스도쿠
- X 가 G 의 모든 *unavoidable sets*와 공통원소를 가진다면, G 는 X 의 유일해이다.

풀이 Solution

- The necessity of 'Remove Duplicates', and 'Add Missing Cases' parts :
 - We are looking for 16-clue puzzles.
 - Suppose we have selected 15 clues so far, and the last unavoidable set is 2.
 - Then no computation is needed (multiple solutions exist)

컴퓨터 계산 Computing

- Previous amount of computation

$$\frac{6.7 \times 10^{21} \times 3.4 \times 10^{16}}{\text{The number of } 9 \times 9 \text{ Sudoku grids} \quad \text{The ways to choose 16 cells out of } 9 \times 9 \text{ grid}}$$

- Current amount of computation

$$\frac{5.47 \times 10^9 \times 9.07 \times 10^{13}}{800 \text{ process-years}}$$

결론 Conclusion

- This search showed that no set of 16 clues can uniquely determine a Sudoku solution.
- 17 is the minimum number of clues that determines the unique puzzle completion.

							1	
					2			3
			4					
						5		
4		1	6					
		7	1					
	5					2		
				8			4	
	3		9	1				

결론 Conclusion

- How mathematics is used in computer science
 - Discrete Mathematics (graph, combinatorics)
 - Linear Algebra (eigenvalue, transformation)
 - Probability & Statics
- Make the algorithms more efficient

Reference

- McGuire, G., Tugemann, B., & Civario, G. (2014). *There is no 16-clue Sudoku: Solving the Sudoku minimum number of clues problem via hitting set enumeration. Experimental Mathematics, 23(2), 190–217.*