SIMPLE HOMOTOPY

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Introduction

Conjecture 1 (Zeeman's conjecture 1964 –). *For any contractible 2-dim CW-complex K, $K \times [0, 1]$ is collapsible.*

This conjecture implies the Poincare conjecture.

Conjecture 2 (Poincare's conjecture 1904 – 2002). *Every closed, simply- connected 3-dim manifold is homeomorphic to S^3.*

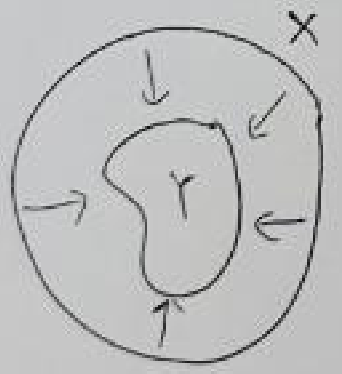
1. Preliminaries

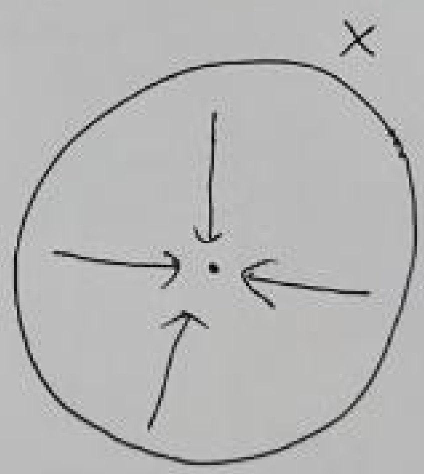
Let us assume that the functions between spaces are all continuous.

Definition 1.1. $f, g : X \to Y$ are homotopic if there exist a function $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Definition 1.2. $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$ such that $gf \cong Id_X$ and $fg \cong Id_Y$. If there exist a homotopy equivalence between X and Y, we call X and Y are homotopy equivalent. If X is homotopy equivalent to a point, then X is contractible.

Example 1.3. Strong deformation retraction $Y \searrow X$





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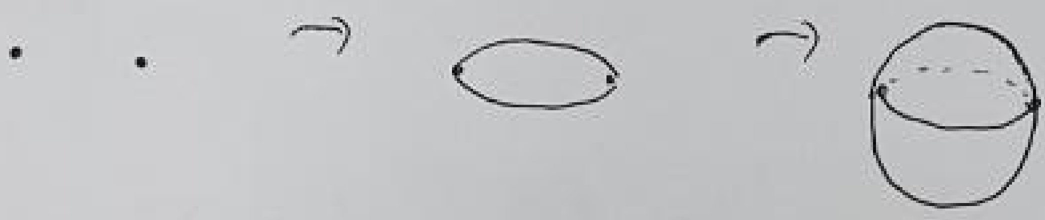
Definition 1.4 (CW-complex). $\varphi_\alpha : I^n \to K$ is a n-cell which satisfies

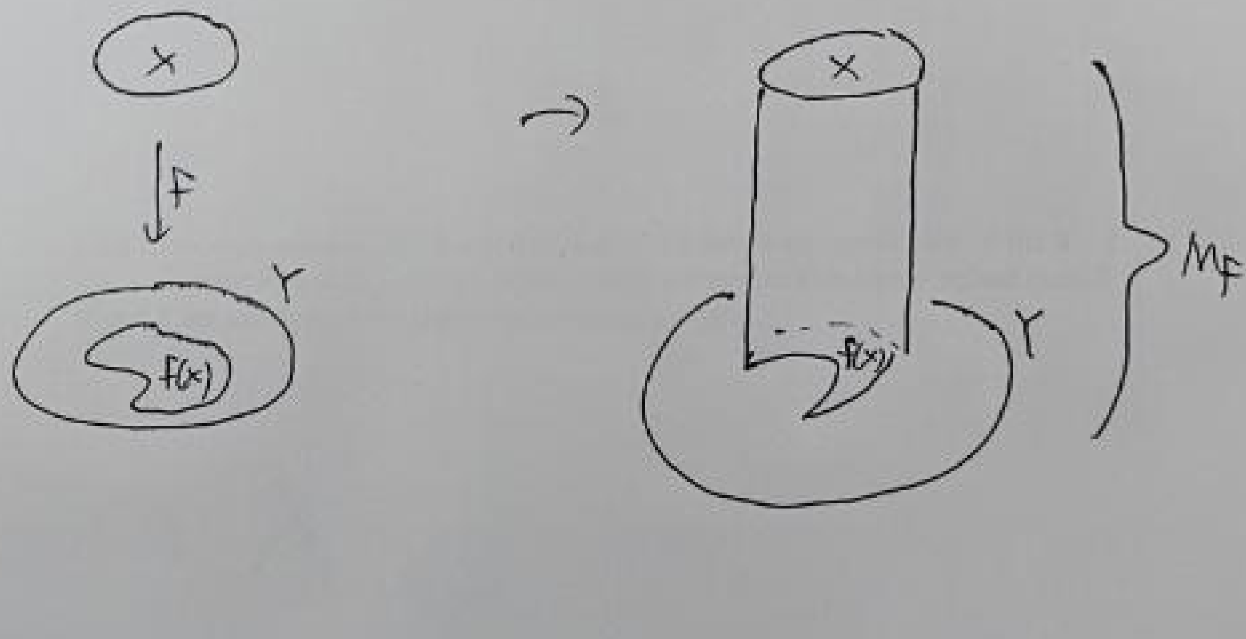
(1) $\varphi_\alpha | \mathring{I}^n$ is a homeomorphism onto e_α
(2) $\varphi_\alpha(\partial I^n) \subset K^{n-1}$

Example 1.5. S^2 can be constructed with two 0-cells, two 1-cells and two 2-cells.

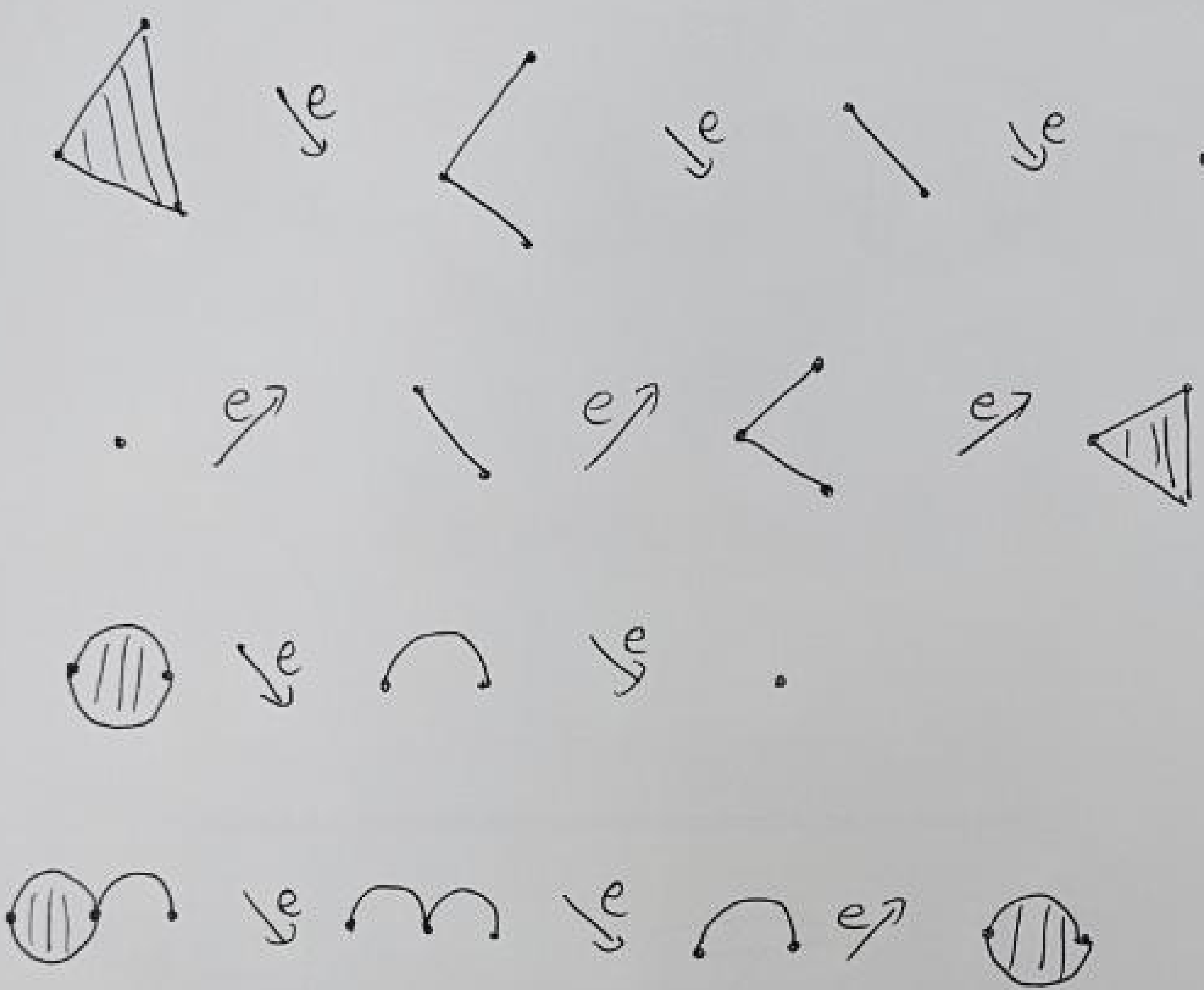


Definition 1.6. Let $f : X \to Y$. Then the mapping cylinder M_f is defined by

$$M_f = (X \times I) \oplus Y / (x, 1) = f(x)$$



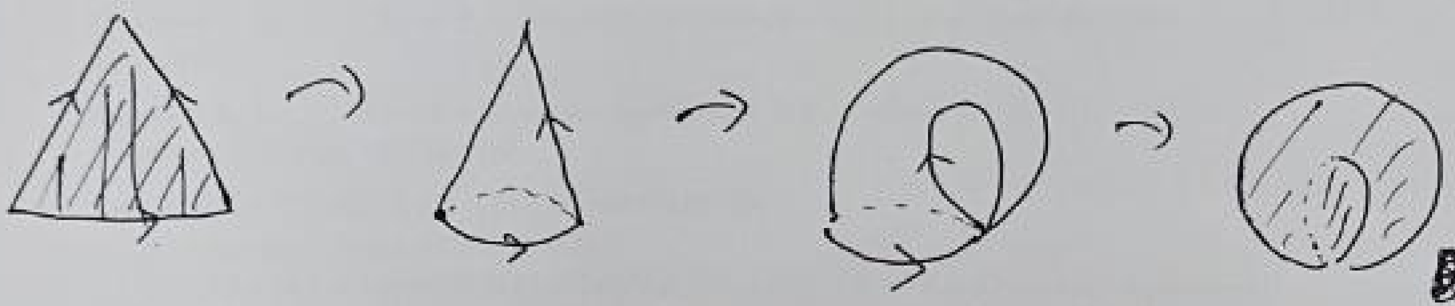
2. Collapse and Expand



Each elementary collapse and elementary expand is a homotopy equivalence. Write K L if L is obtained by a finite sequence of elementary collapse or elementary expand from K. K and L are said to have the same simple-homotopy type.

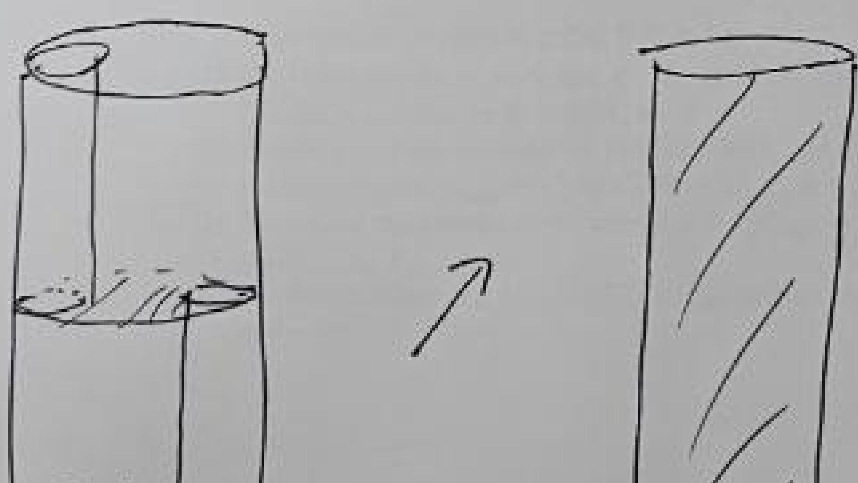
~~Note) elementary collapse/expand~~

Example 2.1. Dunce hat is contractible, but not collapsible.

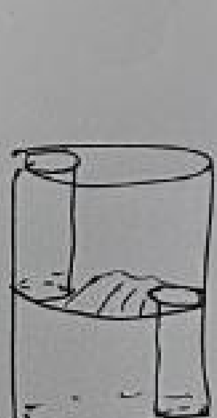


$\simeq B^3 \simeq pt$

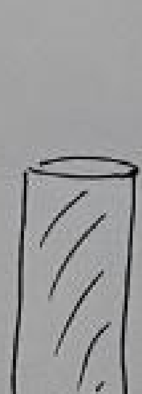
Example 2.2. The house with two rooms is contractible, but not collapsible.



$D^2 \times I \searrow pt$



$\simeq$



$\simeq pt$

Example 2.3 (Zeeman's conjecture). For every finite contractible 2-dim CW-complex K, $K \times [0, 1]$ is collapsible.

Question 1. *Let $f : K \to L$ a homotopy equivalence. Is f also a simple-homotopy equivalence?*

Question 2. *Let $f : K \to L$ a homotopy equivalence. Is there a map $g : K \to L$ which is a simple-homotopy equivalence?*

Both are generally false, but it is true for some conditions.

(1) Question 1 is true iff $Wh(L) = 0$.
(2) Question 2 is true iff $Wh(L) = \{\tau(f) \in Wh(L) \mid f : L \to L \text{ is a homotopy equivalence}\}$

It is known when ($\pi_L = 0$, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, $\mathbb{Z}_p(p = 1, 2, 3, , 4, 6)$, Haken manifolds), then $Wh(L) = 0$. Also for every finite connected 2-complexes L with $\pi_1(L) = \mathbb{Z}_p(p \neq 1, 2, 3, 4, 6)$, Question 2 is true, while Question 1 is not true.

3. Whitehead Group

Let $K \quad L$ and $K' \quad L$. Define $(K, L) \sim (K'L)$ iff $K \quad K'$ *rel* L. Denote $[K, L]$ the equivalence class of (K, L). Define $\left[K_1, L\right] + \left[K_2, L\right] = \left[K \cup_L K', L\right]$. Then

$$Wh(L) := \{[K, L] \mid K \quad L\}$$

is called the Whitehead group of L, which is a well-defined abelian group. Note that if $K \quad L$ *rel* L, then $[K, L] = [L, L] \in Wh(L)$ is the identity element of the group. Also, $\left[2M_D, L\right]$ is the inverse element of $[K, L]$. Now, define the torsion

$$\tau(f) = \left[M_f \cup_K M_f, L_2\right] \in Wh(L)$$

It is hard to deal with $\tau(f)$ of this definition, so we use the fact below:

Theorem 3.1. *For every finite CW complex L, define $T_L : Wh(L) \to Wh(\pi_1 L)$, by $T_L([K, L]) = \tau(K, L)$. Then $T = \{T_L\}$ is a natural equivalence of functors.*

Brief construction of the abelian group $Wh(\pi_1 L)$:

(1) For CW complex K such that $K \quad L$, make the chain complex of $C(\tilde{K}, \tilde{L})$, which is a $\mathbb{Z}(G)$-module for $G = \pi_1(L, x)$
(2) Gather the chain complex of modules with odd dimension and even dimension respectively, i.e. $C_{odd} = C_1 \oplus C_3 \oplus \cdots$, $C_{even} = C_0 \oplus C_2 \oplus \cdots$
(3) Determine the torsion of the map $d + \delta : C_{odd} \to C_{even}$, which is $\tau(d + \delta) \in := GL(\mathbb{Z}(G))/E_G$
(4) We denote $Wh(\pi_1 L) := GL(\mathbb{Z}(G))/E_G$, and the elements are $\tau(K, L) = \tau(C(\tilde{K}, \tilde{L})) = \tau(d + \delta)$