Dimension in Mathematics

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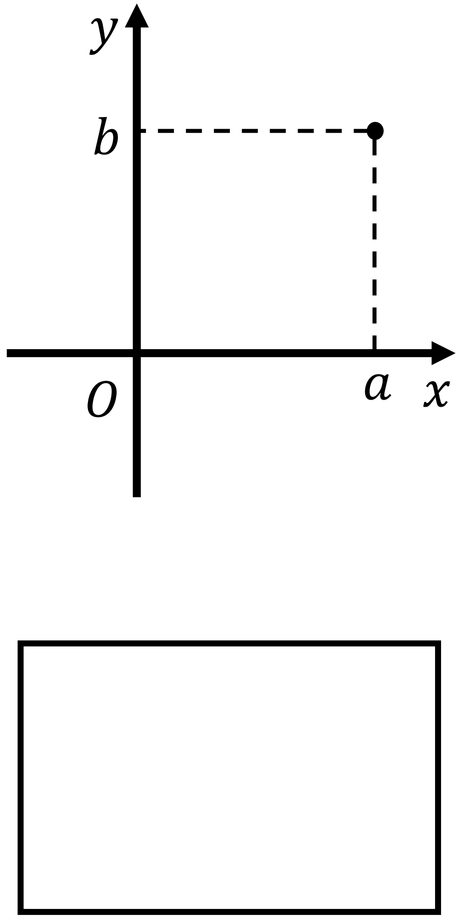
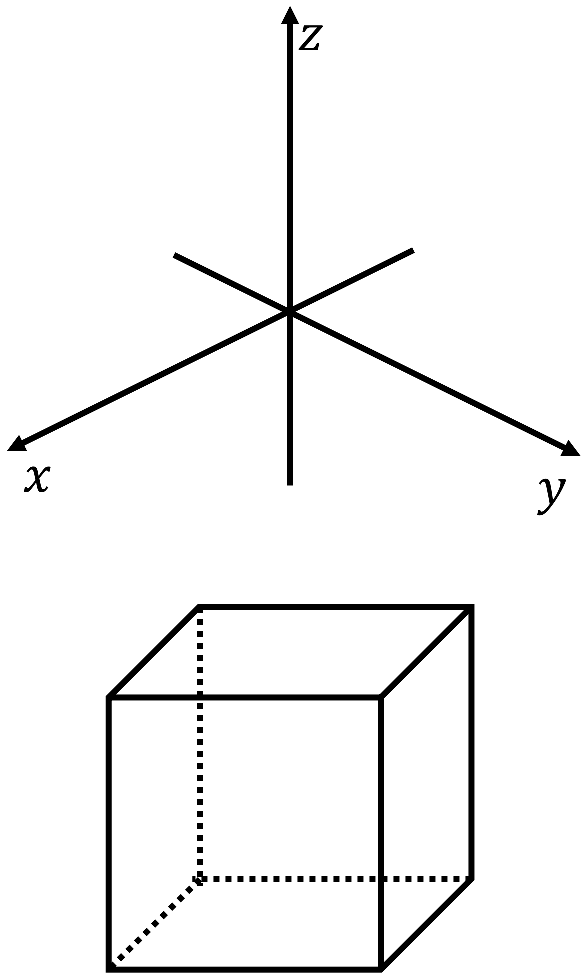
Dimension?

Informally, it is defined as the minimum number
of coordinates needed to specify any point in a space.

Alternatively, it can be considered as the degrees of freedom of
movement within the space.



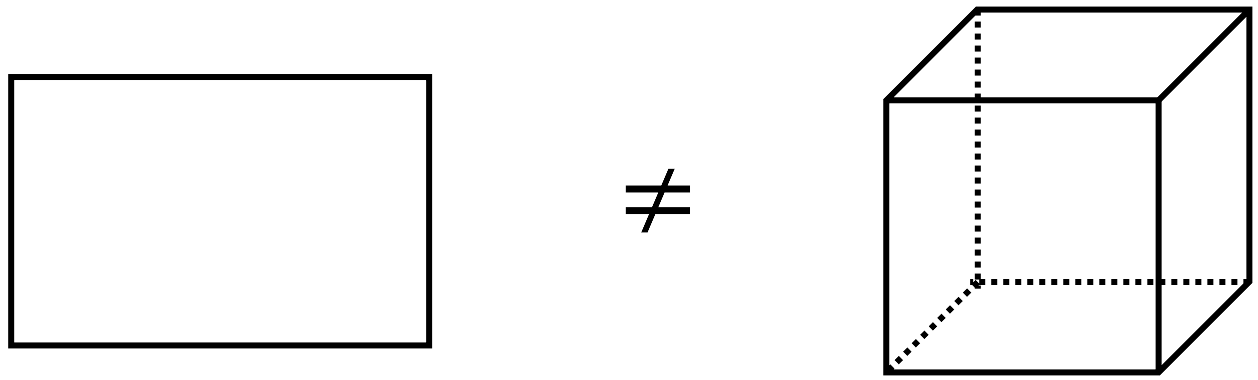
2-dimension



3-dimension

Property of the dimension we already know

If $m \neq n$, then an m-dimensional figure is different from an n-dimensional figure.



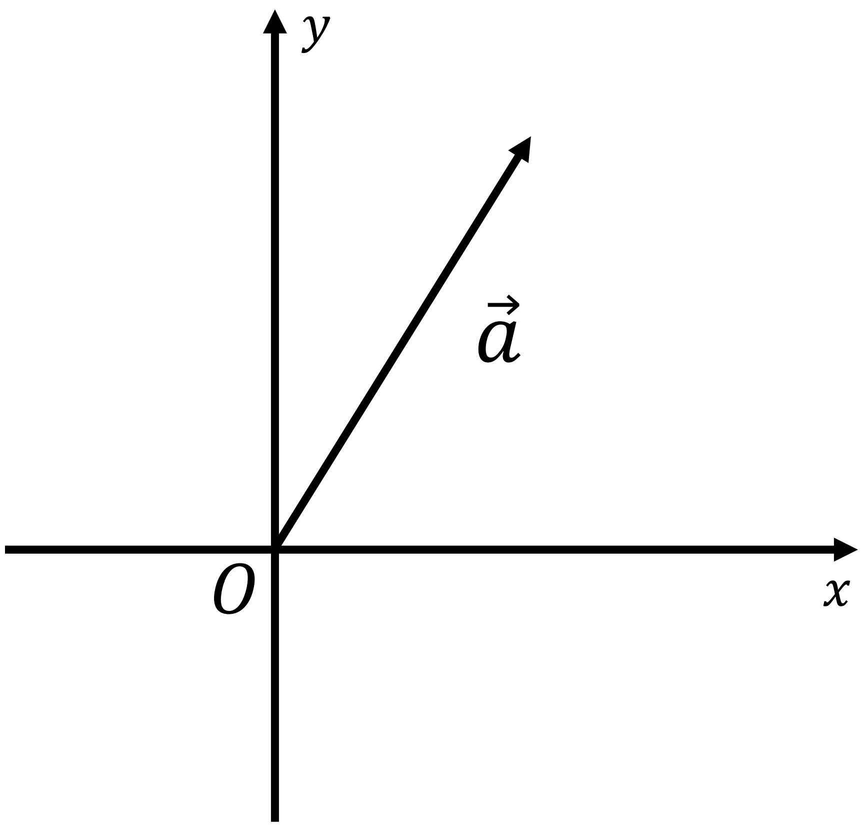
Dimension in mathematics

To use this elementary but useful property, mathematicians have worked to define dimension in various areas of mathematics such as linear algebra, topology, fractal geometry, algebra, etc.

Thus, we’re going to see how it is defined in each subjects.

Dimension in Linear algebra

Vector space



Informally, a vector space is a set of vectors.

$\mathbb{R}^2$ is an example of vector space.

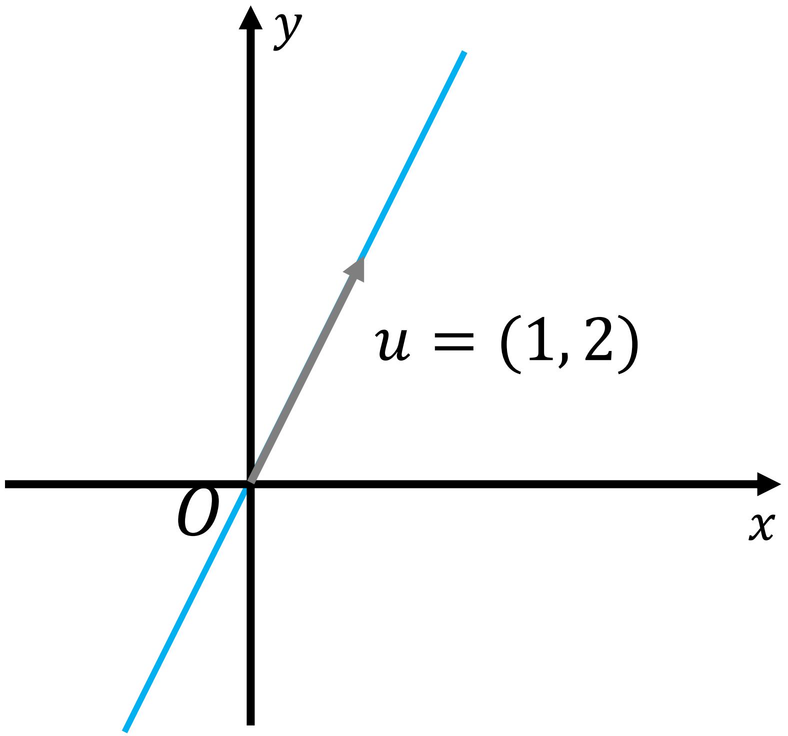
Vector space

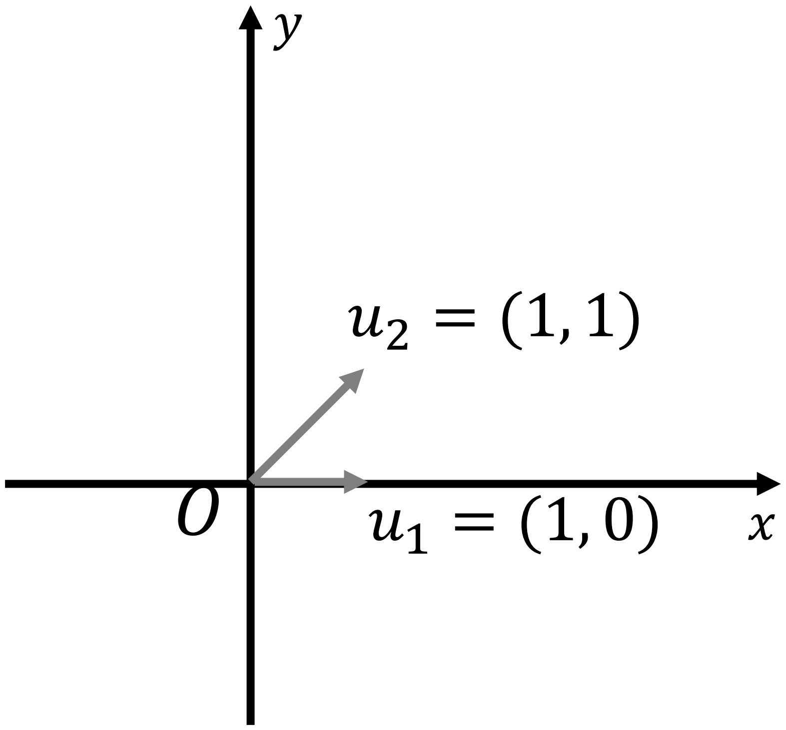
In general, there are infinitely many vectors in a vector space,

so it’s hard to analyze vector spaces without any criteria.

Thus, we introduce a “criterion” for vector spaces.

The criterion have to express “all” the vectors “uniquely”.

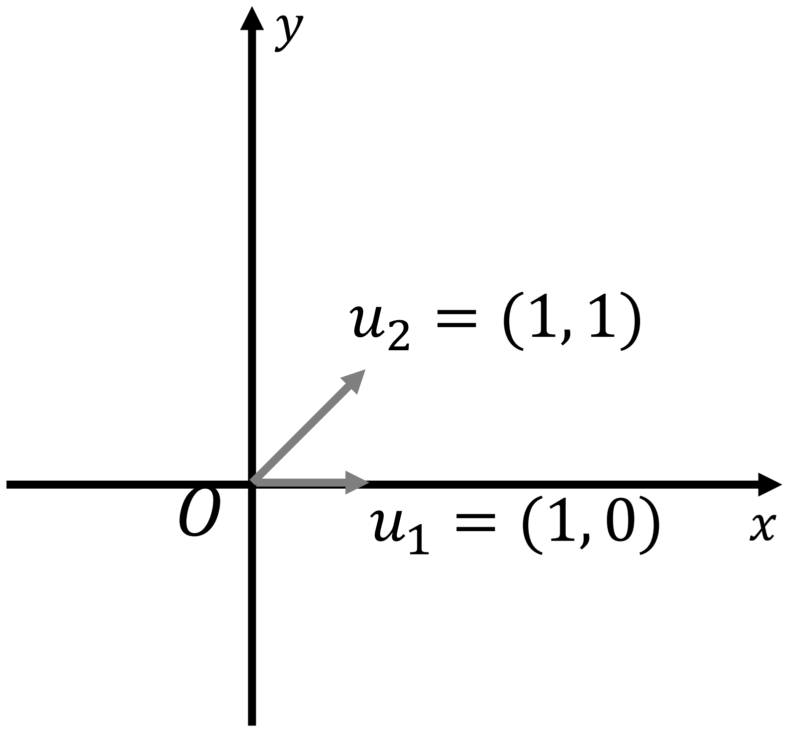


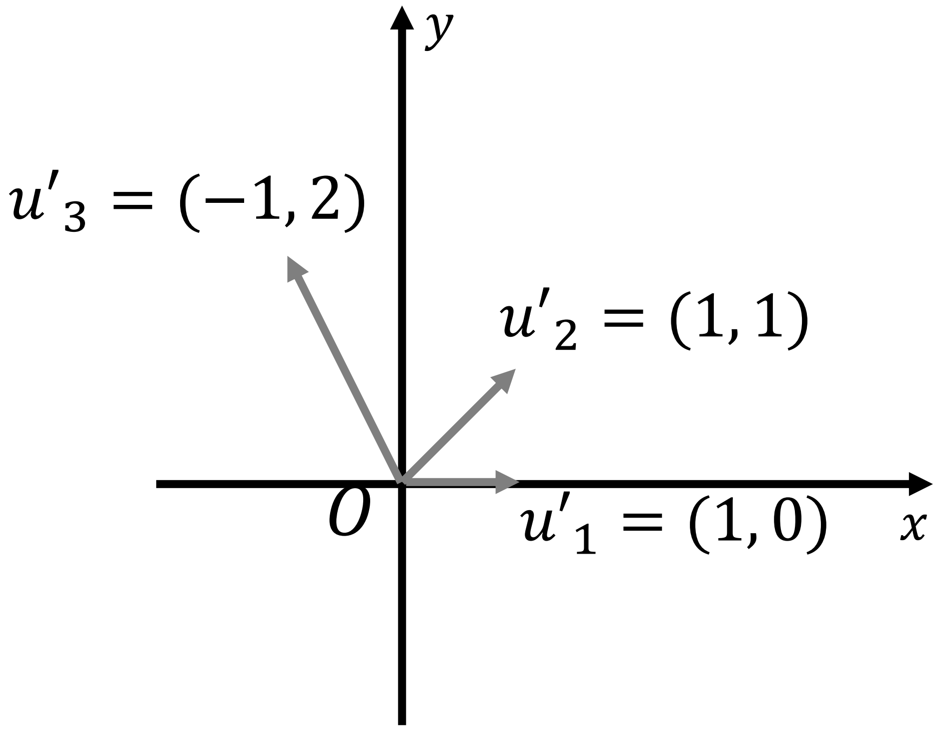


$Span(\{u\}) = \{(t, 2t) \mid t \in \mathbb{R}\}$

Since any $(a, b) = (a - b)u_1 + bu_2$,

$$Span(\underbrace{\{u_1, u_2\}}_{\textbf{Spanning set}}) = \mathbb{R}^2$$





No u_i is expressed by other u_j
→ $\{u_1, u_2\}$ is **linearly independent**

$u'_3 = (-3)u'_1 + 2u'_2$
→ $\{u'_1, u'_2, u'_3\}$ is **linearly dependent**

Basis

A nonempty subset $B \subseteq V$ is a **basis** for V if B spans V and is linearly independent.

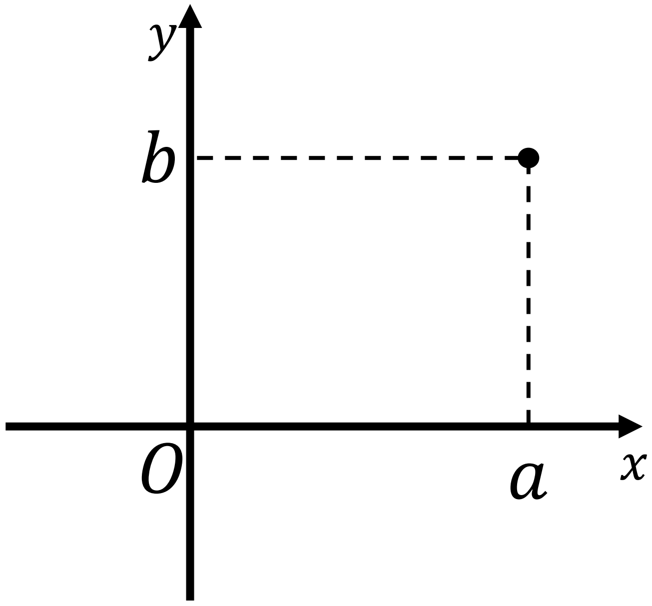
Note that if B and B' are bases for V, then $|B| = |B'|$.

→ It is an invariant of the vector space.

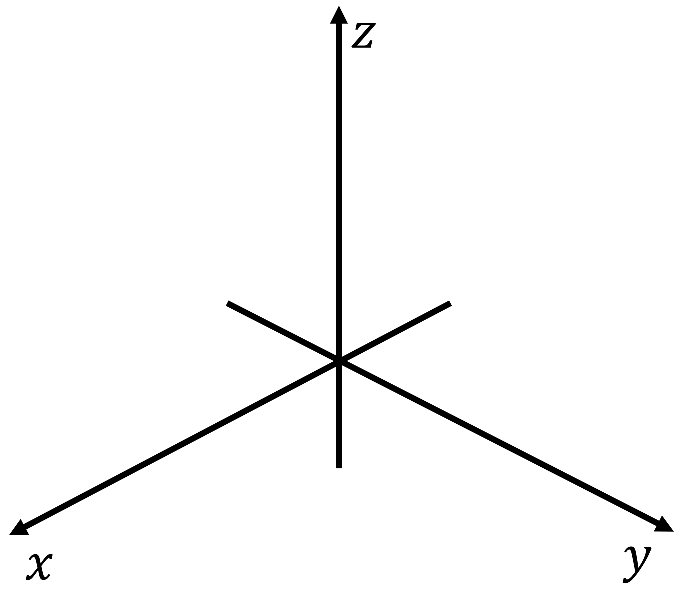
Dimension of vector space

The **dimension** of V, denoted by $\mathbf{dim}\ \boldsymbol{V}$, is defined as the number of elements in any basis of V.

$\dim \mathbb{R} = 1$



$\dim \mathbb{R}^2 = 2$



$\dim \mathbb{R}^3 = 3$

Remark

Let V be a vector space. A subset $W \subseteq V$ is called a **subspace** of V(denoted by $\boldsymbol{W \leq V}$) if $\forall x, y \in W$, $x + y,\ a \cdot x \in W$.

Thm. If $W \leq V$, then $\dim W \leq \dim V$.

Remark

A function $T: V \to W$ is an **isomorphism** if it's a bijective linear transformation.

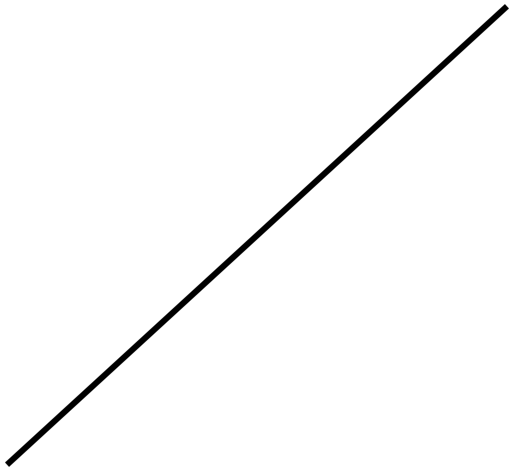
V and W are **isomorphic**($V \cong W$) if $\exists$an isomorphism $T: V \to W$.

Thm. For vector spaces V and W, $V \cong W \iff \dim V = \dim W$.

Dimension of Manifold

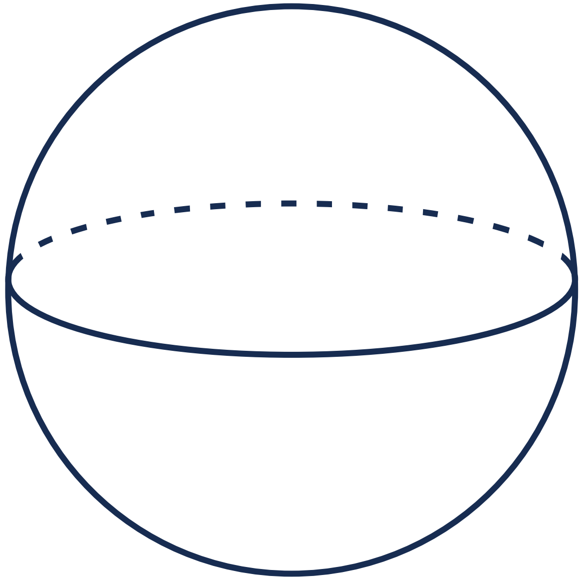
Some problems

Vector spaces are “flat”, i.e., it doesn’t have “curved” region.



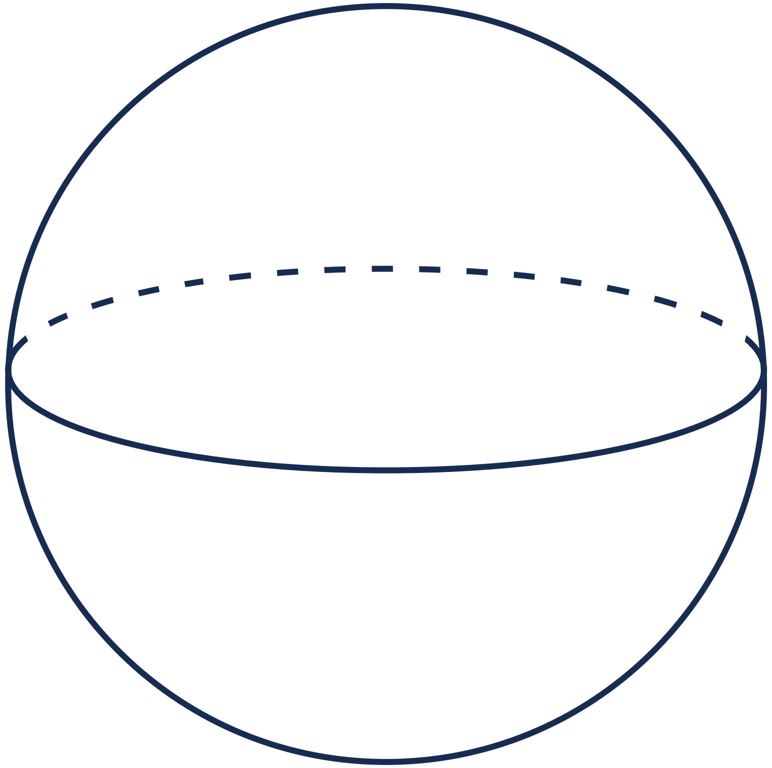
1-dim vector space

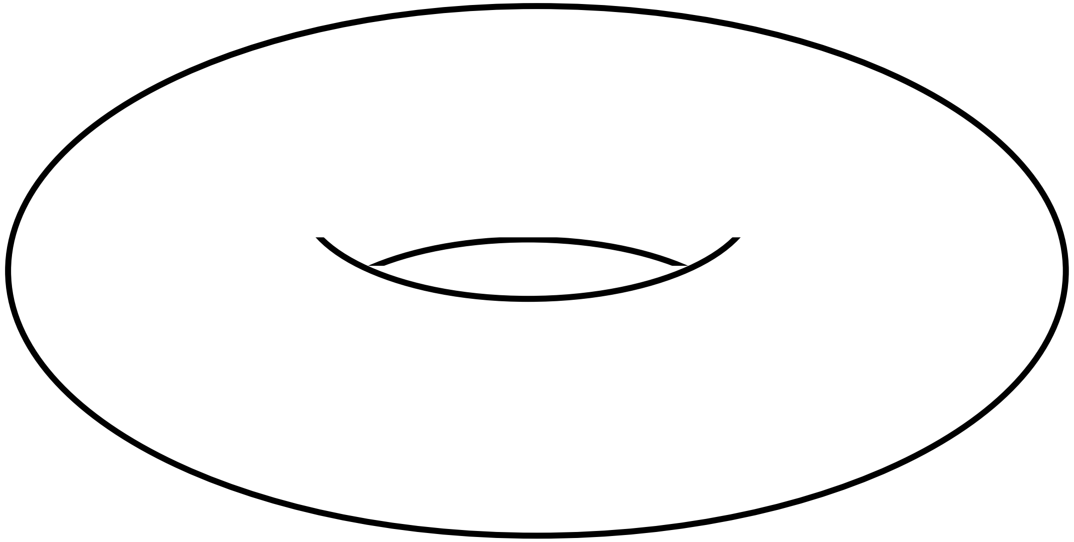
2-dim vector space



not a vector space

What is the dimension of a sphere?





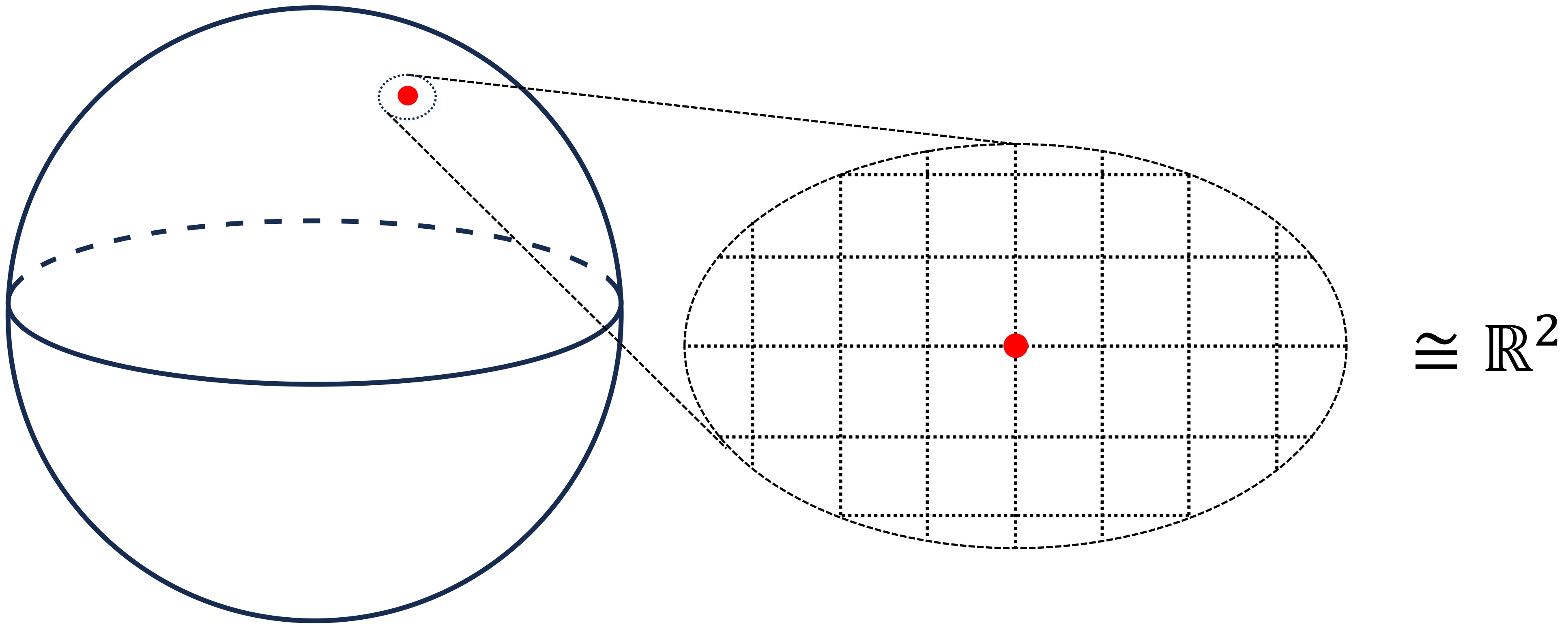
What is the dimension of a sphere?

To define the dimensions of some good figures(circle, sphere, torus. etc.), we have to define “good figure” more precisely, called a **manifold.**

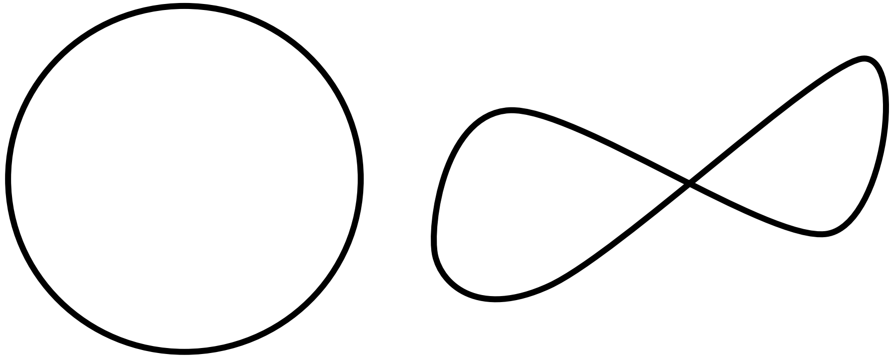
n-dimensional manifold

A topological space M is a **n-dimensional manifold** if

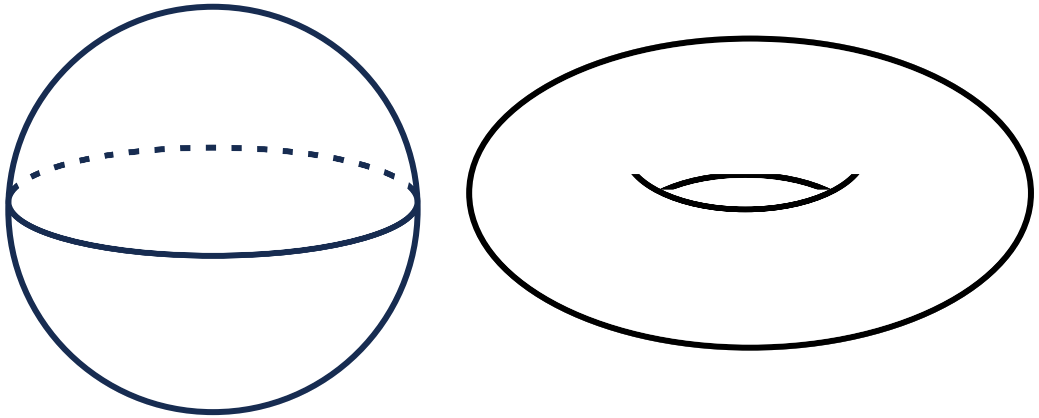
- M is a Hausdorff space
- M is second-countable
- **$\boldsymbol{M}$ is locally Euclidean, i.e., $\forall \boldsymbol{x} \in \boldsymbol{M}$, $\exists \boldsymbol{N_x}$ s.t. $\boldsymbol{N_x}$ is homeomorphic to $\mathbb{R}^{\boldsymbol{n}}$**



Thus, a sphere is a 2-dimensional manifold(or simply, 2-dim figure)



1-dim manifold(curve)



2-dim manifold(surface)

Dimension in Topology

Is $\mathbb{R}^2$(or square($= I^2$)) really dimension 2?

What I mentioned about dimension is that the dimension is the minimum number of coordinates needed to specify any point within it.

For example, we can make a bijection $f: \mathbb{R}^n \to V$ s.t.

$$f(a_1, \dots, a_n) = a_1 u_1 + \cdots + a_n u_n$$

for a n-dim vector space V and a basis $\{u_1, \dots, u_n\} \subseteq V$.

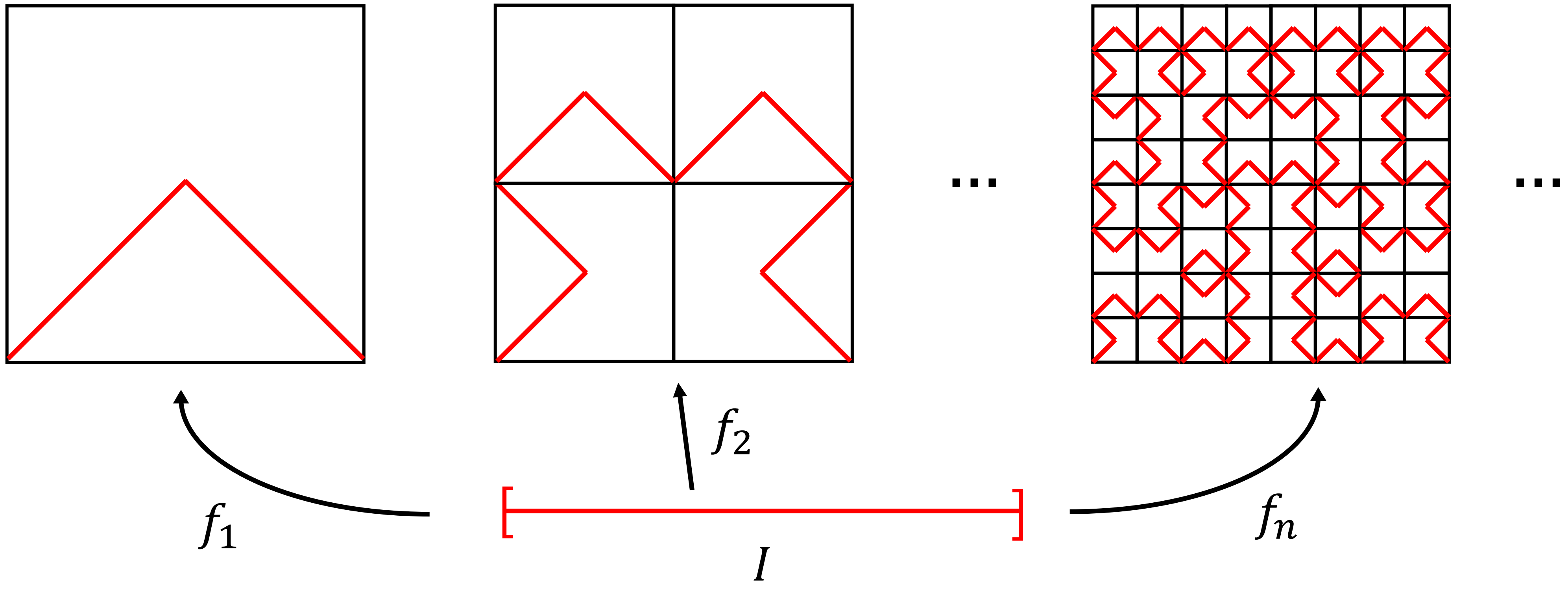
Is $\mathbb{R}^2$(or square($= I^2$)) really dimension 2?

Since $\mathbb{R}$ and $\mathbb{R}^2$ have the same cardinality, there exists a bijection $f: \mathbb{R} \to \mathbb{R}^2$, i.e., we can represent all points of "intuitively 2-dimension" $\mathbb{R}^2$ by just using "intuitively 1-dimension" $\mathbb{R}$.

However, it does not seem reasonable because f may not be continuous(so, $f(x)$ doesn't change continuously while x does).

Space filling curve

There is a surjective continuous function $f\colon I \to I^2$ ($I \coloneqq [0,1]$)



What is the dimension in topology?

By a space filling curve, there is a reasonable function that represents I^2 by a single parameter.

This shows that our informal notion of dimension is insufficient.

What is the dimension in topology?

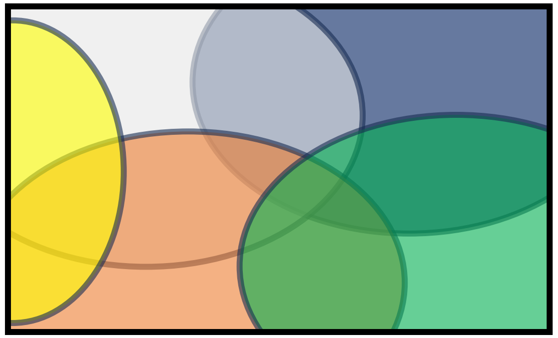
For a topological space, there are just “open sets”, so we can’t define “direction” directly unlike in linear algebra.

Instead, by counting how open sets are overlapping, we can define dimension as in another subjects.

Lebesgue covering dimension

Let X be a topological space.

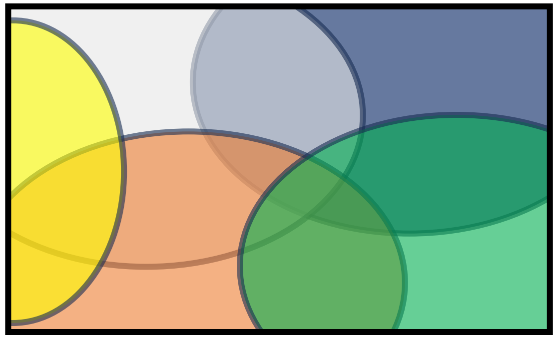
A collection $\mathfrak{A}$ of subsets of X has **order** m if some $x \in X$ lies in m elements of $\mathfrak{A}$, and no point of X lies in more than m elements of $\mathfrak{A}$.



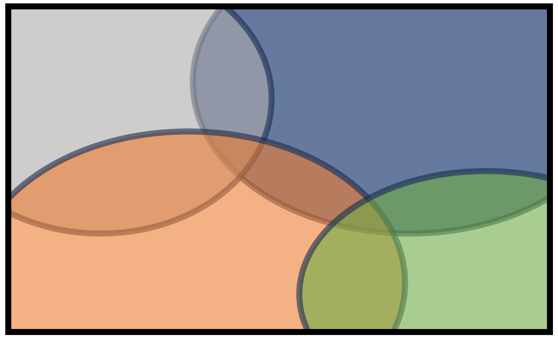
$\leftarrow$ a collection $\mathfrak{A}$ with order 4

Lebesgue covering dimension

A collection $\mathfrak{B}$ **refines**(or is a **refinement** of) a collection $\mathfrak{A}$ if for each $B \in \mathfrak{B}$, there is $A \in \mathfrak{A}$ s.t. $B \subseteq A$.



A collection $\mathfrak{A}$



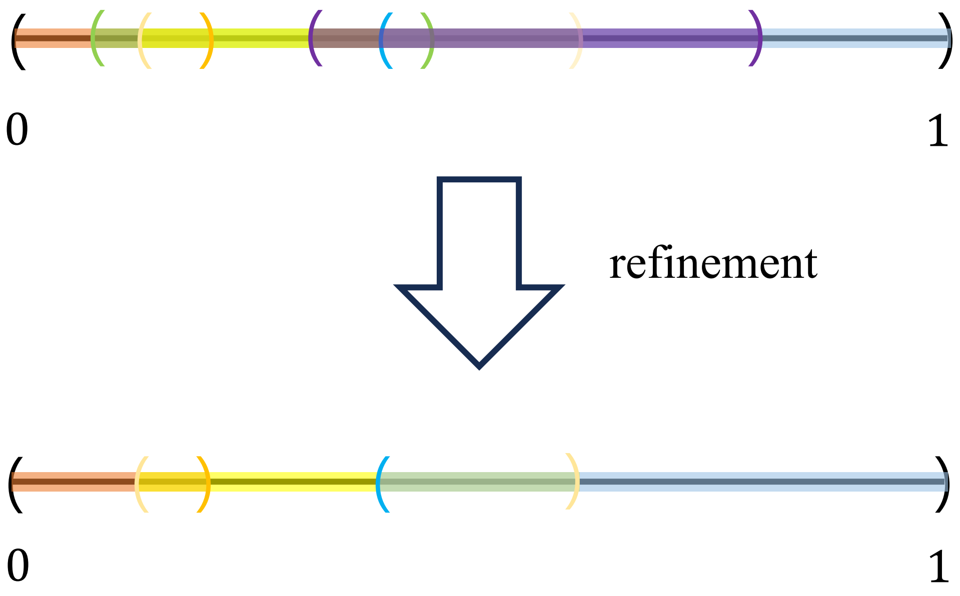
A refinement $\mathfrak{B}$ of $\mathfrak{A}$

order 3

Lebesgue covering dimension

The **Lebesgue covering dimension** of X is defined as the smallest integer m s.t. $\forall$open covering $\mathfrak{A}$ of X, $\exists$open covering $\mathfrak{B}$ of X that refines $\mathfrak{A}$ and has order at most $m + 1$.

Examples



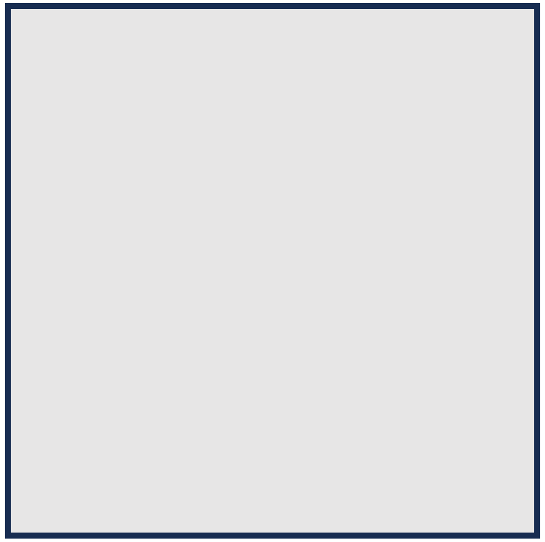
Examples

$$\mathfrak{A} = \left\{ \left(\frac{1}{n+1}, 1 \right) : n \geq 1 \right\}$$

$$\mathfrak{B} = \left\{ \left(\frac{1}{n+1}, \frac{1}{n} + \frac{n-1}{n^3} \right) : n \geq 1 \right\}$$

$\rightarrow$ $(0, 1)$ is 1-dimensional space

Examples



I^2 is 2-dimensional space

Remark

It is possible that the dimension of a subspace can be larger than that of the ambient space.

Let $U = \mathbb{R}^n$ be standard space, and let $X = U \cup \{\infty\}$ with the topology $\mathcal{T}_X = \mathcal{T}_U \cup \{X\}$.

Then $\dim X = 0$ while $\dim U = n$.

Remark

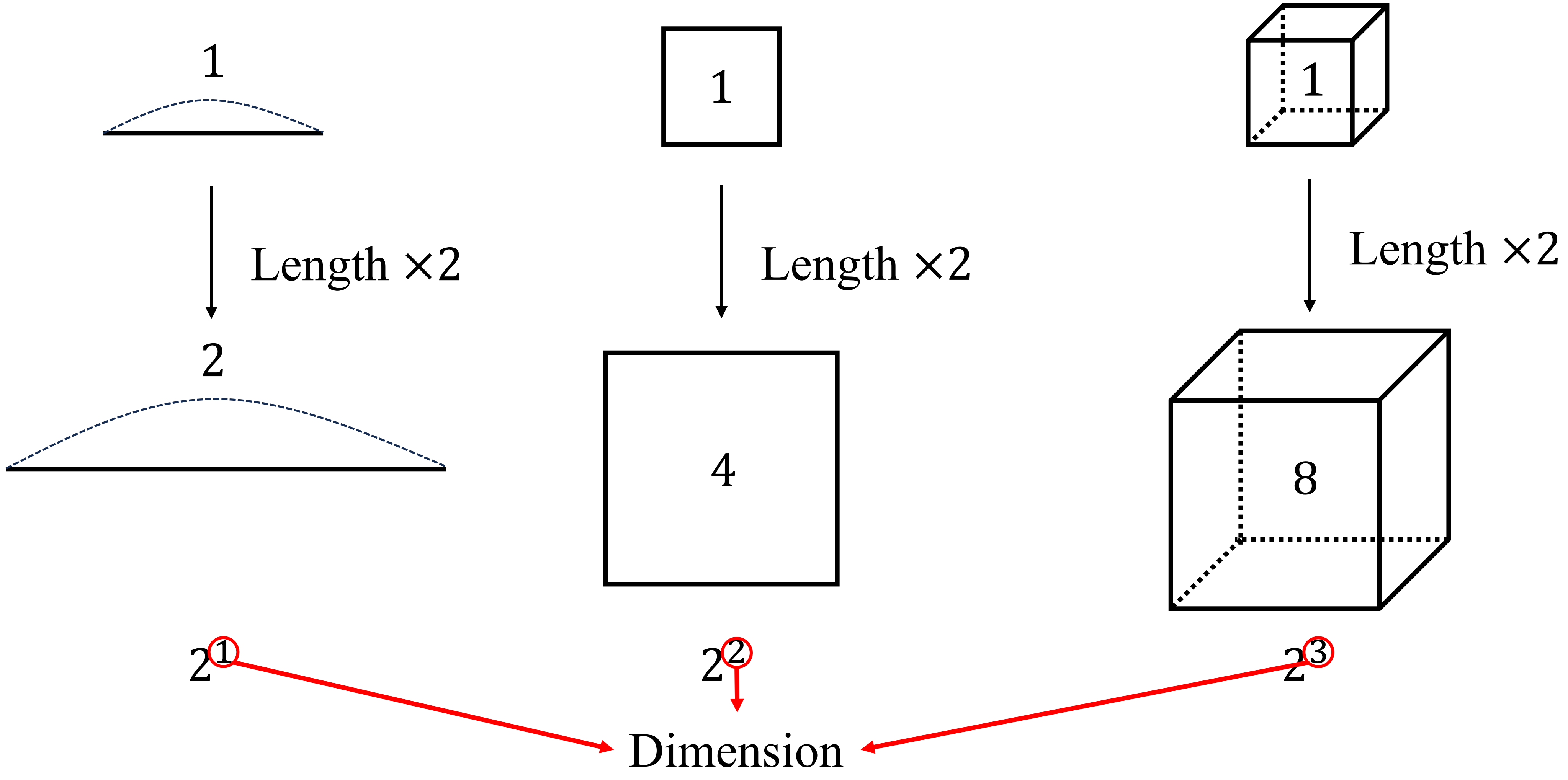
For given topological space X and it's "closed" subspace $C \subseteq X$, $\dim C \leq \dim X$.

Thm. $X \cong Y \implies \dim X = \dim Y$

Cor. $m \neq n \implies \mathbb{R}^m \ncong \mathbb{R}^n$

Hausdorff Dimension

Another aspect of the dimension



Another aspect of the dimension

We can think of the dimension of a figure(or a set) as follow:

If scaling the figure(or the set) by a factor of x multiplies its size(or precisely, measure) by n, then its dimension is defined as $\boldsymbol{\log_x n}$**.**

There is a concept in measure theory that covers above perspective, which is called **Hausdorff dimension**.

Hausdorff dimension

Let X be a metric space and let $\mathrm{S} \subseteq X$ be a subset. Define

$$H_\delta^d(S) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}\, U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, \mathrm{diam}\, U_i < \ \delta\right\}$$

Then, a **d-dimension Hausdorff measure** is defined to be

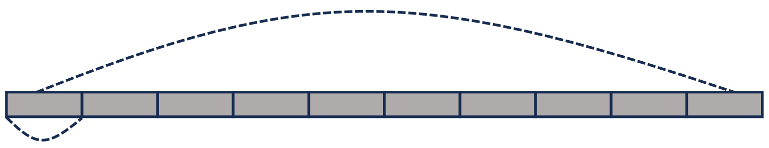
$$H^d(S) \coloneqq \lim_{\delta \to 0} H_\delta^d(S)$$

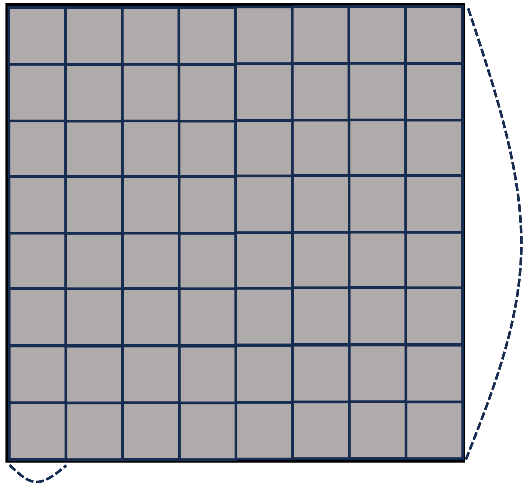
Hausdorff dimension

Note that for fixed subset $S \subseteq X$, as the value of d changes, the value of $H^d(S)$ is one of $\infty(d < \alpha)$, $0(d > \alpha)$, $\alpha'(\in (0, \infty])(d = \alpha)$.

The **Hausdorff dimension** $\dim_H(S) \coloneqq \inf\{d \geq 0 : \ H^d(S) = 0\}$

Examples





$$H^d_{\frac{1}{n}}(I) \le \sum_{i=1}^{n} \left(\frac{1}{n}\right)^d = n^{1-d}$$

$$H^d_{\frac{1}{n}}(I^2) \le \sum_{i=1}^{n^2} \left(\frac{1}{n}\right)^d = n^{2-d}$$

$$H^d(I) = \lim_{n\to\infty} H^d_{\frac{1}{n}}(I) = \begin{cases} \infty, & d < 1 \\ 1, & d = 1 \\ 0, & d > 1 \end{cases}$$

$$H^d(I^2) = \lim_{n\to\infty} H^d_{\frac{1}{n}}(I^2) = \begin{cases} \infty, & d < 2 \\ 1, & d = 2 \\ 0, & d > 2 \end{cases}$$

$\to \dim_H(I) = 1$

$\to \dim_H(I^2) = 2$

Cantor set

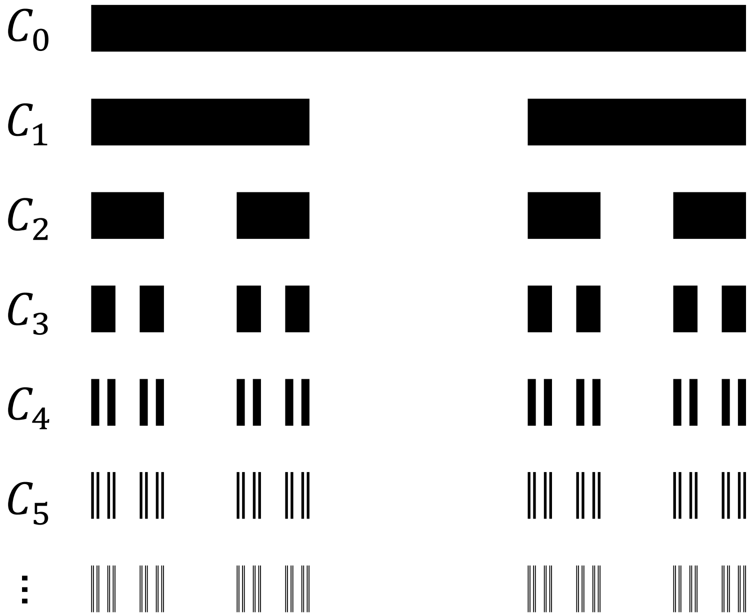
Cantor set

Let $C_0 = [0,1]$, $C_{n+1} = \frac{1}{3}C_n + (\frac{2}{3} + \frac{1}{3}C_n)$. Then Cantor set

$$C = \bigcap_{n=0}^{\infty} C_n$$

Note that each C_n has 2^n segments of length $\frac{1}{3^n}$, and C is uncountable set with length zero.

Hausdorff dimension of Cantor set



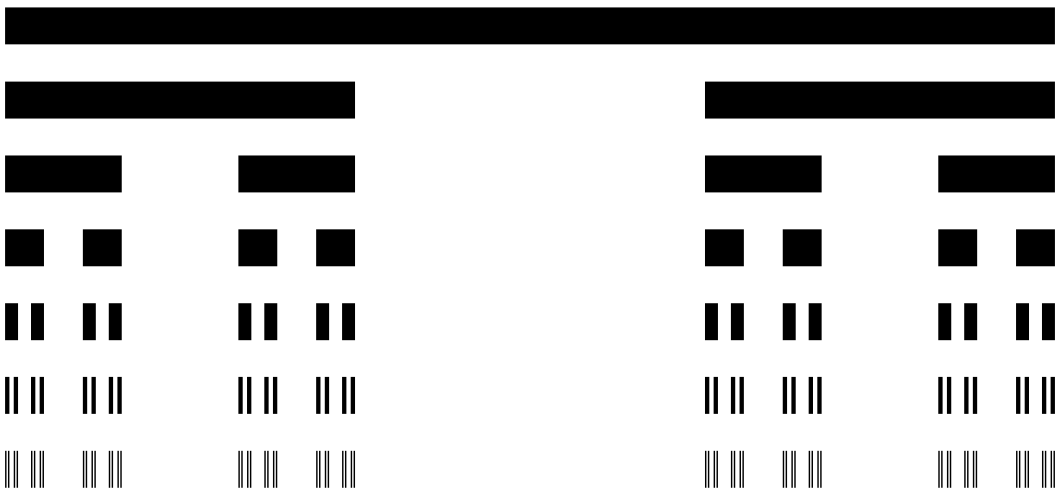
Let $U_{n,\,i}$ be ith segment of C_n. Then

$$H^d_{3^{-n}}(C) \;\le\; \sum_{i=1}^{2^n}\left(diam\left(U_{n,\,i}\right)\right)^d = \sum_{i=1}^{2^n}\left(\frac{1}{3^n}\right)^d = \left(\frac{2}{3^d}\right)^n$$

Thus, $H^d(C) = \lim_{n\to\infty} H^d_{3^{-n}}(C) \le \lim_{n\to\infty}\left(\frac{2}{3^d}\right)^n$, formally,

$$H^d(C) = \begin{cases} \infty, & d < \log_3 2 \\ 1, & d = \log_3 2 \\ 0, & d > \log_3 2 \end{cases} \;\rightarrow\; \dim_H(C) = \log_3 2$$

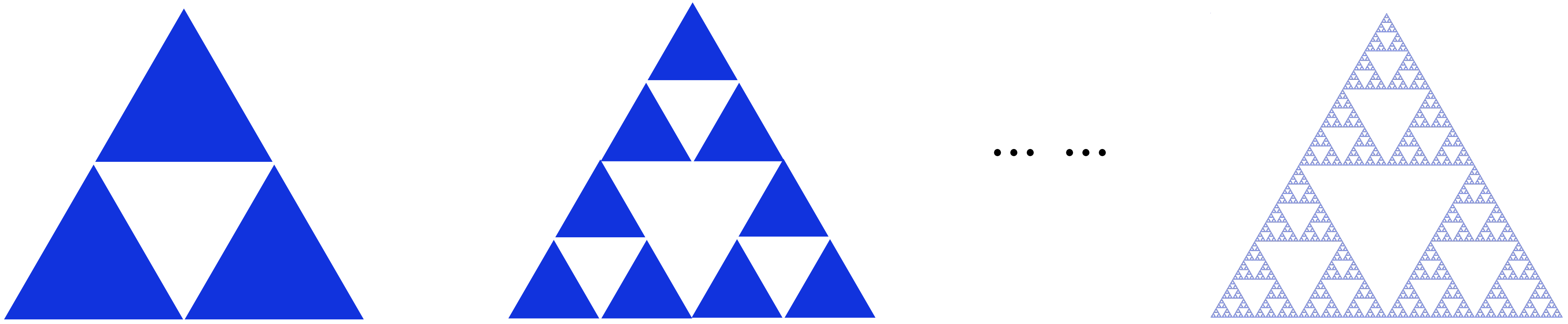
Hausdorff dimension of Cantor set



Since scaling Cantor set by the factor of 1/3 multiplies its size by 1/2,

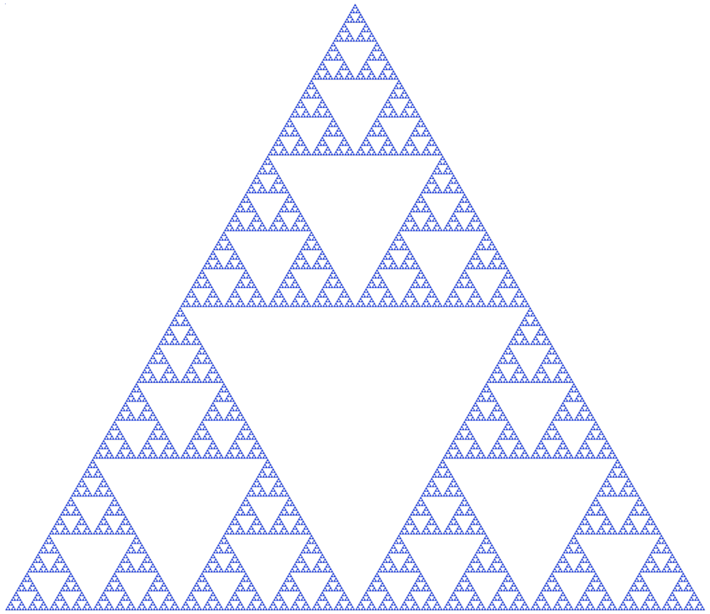
its Hausdorff dimension is $\log_{1/3} 1/2 = \log_3 2$.

Sierpiński triangle



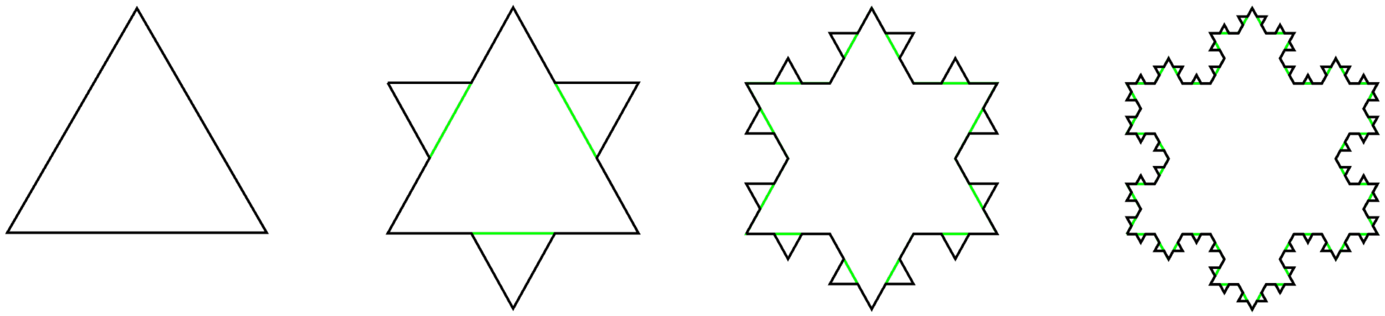
Note that the area of Sierpiński triangle is zero.

Hausdorff dimension of Sierpiński triangle



Likewise, since scaling Sierpiński triangle by the factor of 2 multiplies its size by 3, its Hausdorff dimension is $\log_2 3$.

Koch snowflake



It's Hausdorff dimension is $\log_3 4$.

Bi-Lipschitz Invariant

For two metric spaces (X, d_X), (Y, d_Y), a function $f: X \to Y$ is called **bi-Lipschitz** if $\exists K \geq 1$ s.t. $\forall x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y\big(f(x_1), f(x_2)\big) \leq K d_X(x_1, x_2)$$

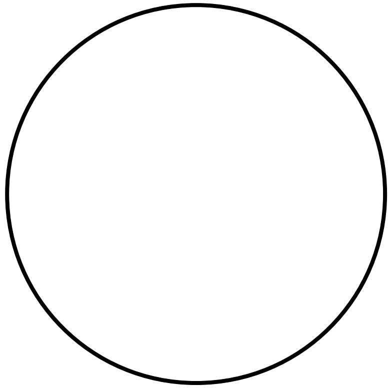
Note that a bi-Lipschitz function is a topological embedding that maps (un)bounded set to (un)bounded set.

Bi-Lipschitz Invariant

(X, d_X) and (Y, d_Y) are **bi-Lipschitz equivalent** if $\exists$a bijective bi-Lipschitz $f: X \to Y$.

Thm. If X and Y are bi-Lipschitz equivalent, then $\dim_H X = \dim_H Y$

Example



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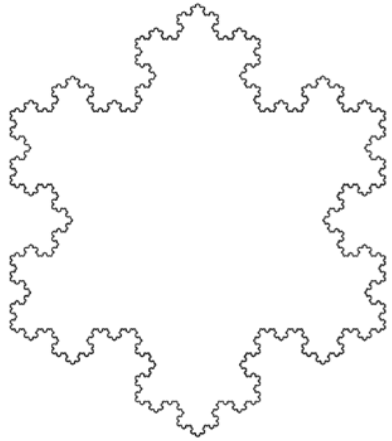

While a circle has the Hausdorff dimension 1, Koch snowflake has $\log_3 4$, so they are not bi-Lipschitz equivalent.

→Hausdorff dimension can classify structures more minutely.

Summary

Dimension

As we saw in each area, in mathematics, one gives some “structure” on a set, and mathematicians want to classify all such things.

Here, “dimension” is a number that represents essential properties(freedom, complexity, etc.) of the spaces.

Thank you!